On the relationship between the thin film equation and Tanners law

Matias G. Delgadino

PUC-Rio de Janeiro

September 9, 2020

Joint work with Antoine Mellet (UMD)

http://bit.ly/windhsielddrop

Thin film equation: model assumptions (Greenspan 78)

Spreading of thin liquid droplets on a planar surface in the complete wetting regime.

Thin film equation: model assumptions (Greenspan 78)

Spreading of thin liquid droplets on a planar surface in the complete wetting regime. i.e. no gravitational forces & perfect hydrophilic surface. (http://bit.ly/HydroPH)

Thin film equation: model assumptions (Greenspan 78)

Spreading of thin liquid droplets on a planar surface in the complete wetting regime. i.e. no gravitational forces & perfect hydrophilic surface. (http://bit.ly/HydroPH)

The droplet shape is given by the graph of a function

$$u(t,x) : (0,\infty) \times \mathbb{R} \ \text{(or } \mathbb{R}^2\text{)} \to \mathbb{R}_+ .$$

Thin film equation: model assumptions (Greenspan 78)

Spreading of thin liquid droplets on a planar surface in the complete wetting regime. i.e. no gravitational forces & perfect hydrophilic surface. (http://bit.ly/HydroPH)

The droplet shape is given by the graph of a function

$$u(t,x) : (0,\infty) \times \mathbb{R} \ \text{(or } \mathbb{R}^2) \to \mathbb{R}_+.$$

Continuity equation

$$\partial_t u + \nabla \cdot (\overline{v} u) = 0,$$

Thin film equation: model assumptions (Greenspan 78)

Spreading of thin liquid droplets on a planar surface in the complete wetting regime. i.e. no gravitational forces & perfect hydrophilic surface. (http://bit.ly/HydroPH)

The droplet shape is given by the graph of a function

$$u(t,x) : (0,\infty) \times \mathbb{R} \ \ (\text{or } \mathbb{R}^2) \to \mathbb{R}_+ .$$

Continuity equation

$$\partial_t u + \nabla \cdot (\overline{v} u) = 0,$$

with

$$\overline{v}(t,x) = \frac{1}{u} \int_0^u v_H(t,x,z)\, dz.$$

Thin film equation: model assumptions (Greenspan 78)

The pressure is independent of height and is given by the mean curvature

$$p(t,x) = -\sigma H_{graph(u)}$$

Thin film equation: model assumptions (Greenspan 78)

The pressure is independent of height and is given by the mean curvature

$$p(t,x) = -\sigma H_{graph(u)} \sim -\sigma \Delta u(t,x) \qquad \text{for flat droplets.}$$

Thin film equation: model assumptions (Greenspan 78)

The pressure is independent of height and is given by the mean curvature

$$p(t,x) = -\sigma H_{graph(u)} \sim -\sigma \Delta u(t,x) \qquad \text{for flat droplets.}$$

Shears in horizontal velocity balances the pressure:

$$\mu \partial_{zz} v_H(t,x,z) = \nabla p(t,x).$$

Thin film equation: model assumptions (Greenspan 78)

The pressure is independent of height and is given by the mean curvature

$$p(t,x) = -\sigma H_{graph(u)} \sim -\sigma \Delta u(t,x) \qquad \text{for flat droplets.}$$

Shears in horizontal velocity balances the pressure:

$$\mu \partial_{zz} v_H(t,x,z) = \nabla p(t,x).$$

Boundary conditions:

$$\begin{cases} \partial_z v_H(t,x,u) & = & 0 & \text{No shear at the stress along the surface} \\ \frac{\Lambda}{u} \partial_z v_H(t,x,0) & = & v_h(t,x,0) & \text{Slip condition.} \end{cases}$$

The coefficient $\Lambda \ll 1$ is of the size of atoms!

Thin film equation: model assumptions (Greenspan 78)

Solving for v_H and

$$\overline{v} = -\frac{\sigma}{\mu}\left(\frac{u^2}{3} + \Lambda u\right)\nabla\Delta u.$$

Thin film equation: model assumptions (Greenspan 78)

Solving for v_H and

$$\overline{v} = -\frac{\sigma}{\mu}\left(\frac{u^2}{3} + \Lambda u\right)\nabla\Delta u.$$

Replacing in the continuity equation

$$\partial_t u + \frac{\sigma}{3\mu}\nabla\cdot((u^3 + 3\Lambda u)\nabla\Delta u) = 0$$

Thin film equation: model assumptions (Greenspan 78)

Solving for v_H and

$$\overline{v} = -\frac{\sigma}{\mu}\left(\frac{u^2}{3} + \Lambda u\right)\nabla\Delta u.$$

Replacing in the continuity equation

$$\partial_t u + \frac{\sigma}{3\mu}\nabla\cdot((u^3 + 3\Lambda u)\nabla\Delta u) = 0$$

Parameters: σ interfacial force, μ viscosity, $\Lambda \ll 1$ slip coefficient.

Thin film equation: model assumptions (Greenspan 78)

Derivation implies the equation should be satisfied only on $\{u > 0\}$.

Thin film equation: model assumptions (Greenspan 78)

Derivation implies the equation should be satisfied only on $\{u > 0\}$.

Simplifying assumption: complete wetting regime, perfectly hydrophilic surface. (http://bit.ly/HydroPH)

Thin film equation: model assumptions (Greenspan 78)

Derivation implies the equation should be satisfied only on $\{u > 0\}$.

Simplifying assumption: complete wetting regime, perfectly hydrophilic surface. (http://bit.ly/HydroPH)

$$\begin{cases} \partial_t u + \frac{\sigma}{3\mu} \nabla \cdot ((u^3 + 3\Lambda u) \nabla \Delta u) = 0 & \text{in } (0, \infty) \times \Omega \\ (u^3 + 3\Lambda u) \nabla \Delta u \cdot \eta_\Omega = 0 & \text{on } (0, \infty) \times \partial\Omega \\ \nabla u \cdot \eta_\Omega = 0 & \text{on } (0, \infty) \times \partial\Omega \\ u(0, x) = u_{in}(x) & \text{in } \Omega \end{cases}$$

Existence: 1-D Bernis-Friedman (JDE 90) and higher-D Grün (CPDE 04).

Thin film equation: model assumptions (Greenspan 78)

Derivation implies the equation should be satisfied only on $\{u > 0\}$.

Simplifying assumption: complete wetting regime, perfectly hydrophilic surface. (http://bit.ly/HydroPH)

$$\begin{cases} \partial_t u + \frac{\sigma}{3\mu}\nabla \cdot ((u^3 + 3\Lambda u)\nabla\Delta u) = 0 & \text{in } (0,\infty)\times\Omega \\ (u^3 + 3\Lambda u)\nabla\Delta u \cdot \eta_\Omega = 0 & \text{on } (0,\infty)\times\partial\Omega \\ \nabla u \cdot \eta_\Omega = 0 & \text{on } (0,\infty)\times\partial\Omega \\ u(0,x) = u_{in}(x) & \text{in } \Omega \end{cases}$$

Existence: 1-D Bernis-Friedman (JDE 90) and higher-D Grün (CPDE 04).

Uniqueness is an open problem!!!

Weak Slippage

General slip conditions:

$$\Lambda u^{n-2}\partial_z v_H(t,x,0) = v_H(t,x,0) \qquad \text{with } n \in (0,3).$$

Weak Slippage

General slip conditions:

$$\Lambda u^{n-2} \partial_z v_H(t, x, 0) = v_H(t, x, 0) \qquad \text{with } n \in (0, 3).$$

$$\partial_t u + \frac{\sigma}{3\mu} \nabla \cdot ((u^3 + 3\Lambda u^n) \nabla \Delta u) = 0$$

Weak Slippage

General slip conditions:

$$\Lambda u^{n-2}\partial_z v_H(t,x,0) = v_H(t,x,0) \qquad \text{with } n \in (0,3).$$

$$\partial_t u + \frac{\sigma}{3\mu}\nabla \cdot ((u^3 + 3\Lambda u^n)\nabla\Delta u) = 0$$

If $\Lambda = 0$ (no slip condition) movement of the contact line leads to infinite dissipation (Huh & Scriven J. Fluid Mechanics 71).

Weak Slippage

General slip conditions:

$$\Lambda u^{n-2}\partial_z v_H(t,x,0) = v_H(t,x,0) \qquad \text{with } n \in (0,3).$$

$$\partial_t u + \frac{\sigma}{3\mu}\nabla \cdot ((u^3 + 3\Lambda u^n)\nabla\Delta u) = 0$$

If $\Lambda = 0$ (no slip condition) movement of the contact line leads to infinite dissipation (Huh & Scriven J. Fluid Mechanics 71).

We will consider

$$\Lambda \sim \varepsilon \ll 1.$$

Weak Slippage

General slip conditions:

$$\Lambda u^{n-2}\partial_z v_H(t,x,0) = v_H(t,x,0) \qquad \text{with } n \in (0,3).$$

$$\partial_t u + \frac{\sigma}{3\mu}\nabla \cdot ((u^3 + 3\Lambda u^n)\nabla\Delta u) = 0$$

If $\Lambda = 0$ (no slip condition) movement of the contact line leads to infinite dissipation (Huh & Scriven J. Fluid Mechanics 71).

We will consider

$$\Lambda \sim \varepsilon \ll 1.$$

Movement of the contact line of order

$$\frac{1}{|\ln \varepsilon|} \qquad \text{(Glasner, Physics of Fluids 03).}$$

Physical regime

We will analyze the long time scales

$$t \sim |\ln \varepsilon|.$$

Physical regime

We will analyze the long time scales

$$t \sim \left|\ln \varepsilon\right|.$$

Formally, we can expect the pressure to have equilibrated faster than this time scale:

Physical regime

We will analyze the long time scales

$$t \sim |\ln \varepsilon|.$$

Formally, we can expect the pressure to have equilibrated faster than this time scale:

$$0 = \nabla p(t, x) = -\sigma \nabla \Delta u.$$

which implies

$$-\Delta u = \lambda(t).$$

Physical regime

We will analyze the long time scales

$$t \sim |\ln \varepsilon|.$$

Formally, we can expect the pressure to have equilibrated faster than this time scale:

$$0 = \nabla p(t, x) = -\sigma \nabla \Delta u.$$

which implies

$$-\Delta u = \lambda(t).$$

In 1-D, we expect a quick relaxation to almost parabolas.

Quasi-Static approximation

In the limit, we expect

$$\begin{cases} -\Delta u = \lambda(t) & \text{in } \{u(t) > 0\}, \\ V = F(|\nabla u|) & \text{on } \partial\{u(t) > 0\}, \end{cases}$$

where V is the velocity of the free boundary. (Glasner-Kim, Inter Free Bound 09)

Quasi-Static approximation

In the limit, we expect

$$\begin{cases} -\Delta u = \lambda(t) & \text{in } \{u(t) > 0\}, \\ V = F(|\nabla u|) & \text{on } \partial\{u(t) > 0\}, \end{cases}$$

where V is the velocity of the free boundary. (Glasner-Kim, Inter Free Bound 09)

Tanner's law (78): The edge velocity of a spreading droplet is approximately proportional to the cube of the slope at the inflection:

Quasi-Static approximation

In the limit, we expect

$$\begin{cases} -\Delta u = \lambda(t) & \text{in } \{u(t) > 0\}, \\ V = F(|\nabla u|) & \text{on } \partial\{u(t) > 0\}, \end{cases}$$

where V is the velocity of the free boundary. (Glasner-Kim, Inter Free Bound 09)

Tanner's law (78): The edge velocity of a spreading droplet is approximately proportional to the cube of the slope at the inflection:

$$F(|\nabla u|) = \alpha |\nabla u|^3.$$

Quasi-Static approximation

In the limit, we expect

$$\begin{cases} -\Delta u = \lambda(t) & \text{in } \{u(t) > 0\}, \\ V = F(|\nabla u|) & \text{on } \partial\{u(t) > 0\}, \end{cases}$$

where V is the velocity of the free boundary. (Glasner-Kim, Inter Free Bound 09)

Tanner's law (78): The edge velocity of a spreading droplet is approximately proportional to the cube of the slope at the inflection:

$$F(|\nabla u|) = \alpha|\nabla u|^3.$$

WARNING: Droplets merge instantaneously!!!
http://bit.ly/MergedDrops

Quasi-Static Approximation

We should notice that we can just follow the wetted domain keeping track of the mass.

Quasi-Static Approximation

We should notice that we can just follow the wetted domain keeping track of the mass.

If we decompose the wetted region into its connected components

$$\{u(t) > 0\} = \bigcup_{i \in I} \Sigma_i(t),$$

Quasi-Static Approximation

We should notice that we can just follow the wetted domain keeping track of the mass.

If we decompose the wetted region into its connected components

$$\{u(t) > 0\} = \bigcup_{i \in I} \Sigma_i(t),$$

then for every $i \in I$, u minimizes

$$\int_{\Sigma_i(t)} |\nabla v|^2 \, dx$$

subject to

$$v \in H_0^1(\Sigma_i(t)) \qquad \& \qquad \int_{\Sigma_i(t)} v \, dx = \int_{\Sigma_i(t)} u \, dx.$$

Quasi-Static Approximation

Hard problem, so we start with 1-D.

Quasi-Static Approximation

Hard problem, so we start with 1-D.

Advantage the Quasi-Static approximation is well posed:

Quasi-Static Approximation

Hard problem, so we start with 1-D.

Advantage the Quasi-Static approximation is well posed: we can construct explicit solutions.

Quasi-Static Approximation

Hard problem, so we start with 1-D.

Advantage the Quasi-Static approximation is well posed: we can construct explicit solutions.

If we take a component of the wetted region $(a(t), b(t))$, then

$$u(x,t) = 6\left(\int_{a(t)}^{b(t)} u(y)\,dy\right)\frac{(b(t)-x)_+(x-a(t))_+}{(b(t)-a(t))^3} \qquad \text{on } (a(t), b(t)).$$

Quasi-Static Approximation

Hard problem, so we start with 1-D.

Advantage the Quasi-Static approximation is well posed: we can construct explicit solutions.

If we take a component of the wetted region $(a(t), b(t))$, then

$$u(x,t) = 6\left(\int_{a(t)}^{b(t)} u(y)\,dy\right)\frac{(b(t)-x)_+(x-a(t))_+}{(b(t)-a(t))^3} \qquad \text{on } (a(t), b(t)).$$

By Tanner's law

$$\dot{a}(t) = -\dot{b}(t) = -72\left(\int_{a(t)}^{b(t)} u(y)\,dy\right)^3 (b(t)-a(t))^{-6}.$$

Quasi-Static Approximation

Hard problem, so we start with 1-D.

Advantage the Quasi-Static approximation is well posed: we can construct explicit solutions.

If we take a component of the wetted region $(a(t), b(t))$, then

$$u(x,t) = 6\left(\int_{a(t)}^{b(t)} u(y)\,dy\right)\frac{(b(t)-x)_+(x-a(t))_+}{(b(t)-a(t))^3} \qquad \text{on } (a(t), b(t)).$$

By Tanner's law

$$\dot{a}(t) = -\dot{b}(t) = -72\left(\int_{a(t)}^{b(t)} u(y)\,dy\right)^3 (b(t)-a(t))^{-6}.$$

We evolve by the ODE until droplets merge.

Back to PDE

We are interested in the limit $\varepsilon \to 0$ of

$$\begin{cases} \partial_t u^\varepsilon + |\ln \varepsilon|((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx})_x = 0 & \text{in } (0, T) \times \Omega \\ u^\varepsilon(0, x) = u_{in}(x) & \text{in } \Omega \\ ((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx} = 0 & \text{on } (0, T) \times \partial\Omega \\ u^\varepsilon_x = 0 & \text{on } (0, T) \times \partial\Omega \end{cases}$$

Back to PDE

We are interested in the limit $\varepsilon \to 0$ of

$$\begin{cases} \partial_t u^\varepsilon + |\ln \varepsilon|((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx})_x = 0 & \text{in } (0,T) \times \Omega \\ u^\varepsilon(0,x) = u_{in}(x) & \text{in } \Omega \\ ((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx} = 0 & \text{on } (0,T) \times \partial\Omega \\ u^\varepsilon_x = 0 & \text{on } (0,T) \times \partial\Omega \end{cases}$$

We expect u^ε to converge to a solution of the 1-D Quasi-Static approximation with Tanner's law.

Re-scaling

We notice that

$$u^{\varepsilon}(t,x) = \varepsilon h^{\varepsilon}(\varepsilon^{7}|\ln \varepsilon| t, \varepsilon x)$$

where

$$\begin{cases} \partial_t h^{\varepsilon} + \partial_x((h^{\varepsilon n} + h^{\varepsilon 3})\partial_{xxx} h^{\varepsilon}) = 0 \\ h^{\varepsilon}_{in}(x) = \frac{1}{\varepsilon} u_{in}(\frac{x}{\varepsilon}). \end{cases}$$

Re-scaling

We notice that

$$u^{\varepsilon}(t,x) = \varepsilon h^{\varepsilon}(\varepsilon^{7}|\ln\varepsilon|t, \varepsilon x)$$

where

$$\begin{cases} \partial_t h^{\varepsilon} + \partial_x((h^{\varepsilon n} + h^{\varepsilon 3})\partial_{xxx} h^{\varepsilon}) = 0 \\ h^{\varepsilon}_{in}(x) = \frac{1}{\varepsilon} u_{in}(\frac{x}{\varepsilon}). \end{cases}$$

Mathematicians that studied similar equations include Bernis, Bertozzi, Carrillo, Dal Passo, Fischer, Giacomelli, Glasner, Gnann, Knüpfer, Otto, Majdoub, Masmoudi, Matthes, Mellet, Pugh, Savare, Tayachi, Toscani and many more...

Results for the homogeneous equation

Most research done on the homogeneous problem:

$$\partial_t h + \partial_x(h^n \partial_{xxx} h) = 0.$$

Results for the homogeneous equation

Most research done on the homogeneous problem:

$$\partial_t h + \partial_x (h^n \partial_{xxx} h) = 0.$$

For $n \geq 4$, finite entropy solutions need to have full support. (Bertozzi-Pugh, CPAM 96)

For $n < 3$, there is existence of source type solutions.

Results for the homogeneous equation

Most research done on the homogeneous problem:

$$\partial_t h + \partial_x (h^n \partial_{xxx} h) = 0.$$

For $n \geq 4$, finite entropy solutions need to have full support. (Bertozzi-Pugh, CPAM 96)

For $n < 3$, there is existence of source type solutions. In the case $n \geq 3$, the support is not expected to move; Motion of the contact line implies infinite dissipation.

Results for the homogeneous equation

Most research done on the homogeneous problem:

$$\partial_t h + \partial_x (h^n \partial_{xxx} h) = 0.$$

For $n \geq 4$, finite entropy solutions need to have full support. (Bertozzi-Pugh, CPAM 96)

For $n < 3$, there is existence of source type solutions. In the case $n \geq 3$, the support is not expected to move; Motion of the contact line implies infinite dissipation. Source type solutions do not exists for $n = 3$ (Bernis, Peletier, Williams, JDE 92).

Results for the homogeneous equation

Most research done on the homogeneous problem:

$$\partial_t h + \partial_x (h^n \partial_{xxx} h) = 0.$$

For $n \geq 4$, finite entropy solutions need to have full support. (Bertozzi-Pugh, CPAM 96)

For $n < 3$, there is existence of source type solutions. In the case $n \geq 3$, the support is not expected to move; Motion of the contact line implies infinite dissipation. Source type solutions do not exists for $n = 3$ (Bernis, Peletier, Williams, JDE 92).

Finite speed of propagation of the support, (Bernis, Acad. Sci. Paris 96)

Results for the homogeneous equation

Most research done on the homogeneous problem:

$$\partial_t h + \partial_x(h^n \partial_{xxx} h) = 0.$$

For $n \geq 4$, finite entropy solutions need to have full support. (Bertozzi-Pugh, CPAM 96)

For $n < 3$, there is existence of source type solutions. In the case $n \geq 3$, the support is not expected to move; Motion of the contact line implies infinite dissipation. Source type solutions do not exists for $n = 3$ (Bernis, Peletier, Williams, JDE 92).

Finite speed of propagation of the support, (Bernis, Acad. Sci. Paris 96)

Uniqueness result for source type solutions for $n = 1$ (Majdoub, Masmoudi, Tayachi, AMS 18).

Results for the homogeneous equation

Most research done on the homogeneous problem:

$$\partial_t h + \partial_x (h^n \partial_{xxx} h) = 0.$$

For $n \geq 4$, finite entropy solutions need to have full support. (Bertozzi-Pugh, CPAM 96)

For $n < 3$, there is existence of source type solutions. In the case $n \geq 3$, the support is not expected to move; Motion of the contact line implies infinite dissipation. Source type solutions do not exists for $n = 3$ (Bernis, Peletier, Williams, JDE 92).

Finite speed of propagation of the support, (Bernis, Acad. Sci. Paris 96)

Uniqueness result for source type solutions for $n = 1$ (Majdoub, Masmoudi, Tayachi, AMS 18).

Non-Homogeneous: Travelling Wave

There exists travelling wave solutions for

$$\partial_t h + \partial_x((h^n + h^3)\partial_{xxx} h) = 0,$$

with $n \in (3/2, 7/3)$. (Giacomelli-Gnann-Otto, Nonlinearity 16)

Non-Homogeneous: Travelling Wave

There exists travelling wave solutions for

$$\partial_t h + \partial_x((h^n + h^3)\partial_{xxx} h) = 0,$$

with $n \in (3/2, 7/3)$. (Giacomelli-Gnann-Otto, Nonlinearity 16)

That is to say, for every $V > 0$ there exists $H_V : \mathbb{R} \to \mathbb{R}$, such that

$$h(t, x) = H_V(x - Vt)$$

is a solution to the equation.

Non-Homogeneous: Travelling Wave

There exists travelling wave solutions for

$$\partial_t h + \partial_x((h^n + h^3)\partial_{xxx} h) = 0,$$

with $n \in (3/2, 7/3)$. (Giacomelli-Gnann-Otto, Nonlinearity 16)

That is to say, for every $V > 0$ there exists $H_V : \mathbb{R} \to \mathbb{R}$, such that

$$h(t, x) = H_V(x - Vt)$$

is a solution to the equation.

Non-Homogeneous: Travelling Wave

We have

$$H_V(x) \sim x_+(V \ln x)^{1/3} \qquad \text{for } x \to \infty$$

Non-Homogeneous: Travelling Wave

We have

$$H_V(x) \sim x_+(V \ln x)^{1/3} \qquad \text{for } x \to \infty$$

and H_V satisfies Tanner's law up to a logarithm

$$(H_V'(x))^3 \sim V(\ln x) \qquad \text{as } x \to \infty.$$

Non-Homogeneous: Travelling Wave

We have

$$H_V(x) \sim x_+(V \ln x)^{1/3} \qquad \text{for } x \to \infty$$

and H_V satisfies Tanner's law up to a logarithm

$$(H_V'(x))^3 \sim V(\ln x) \qquad \text{as } x \to \infty.$$

Under our re-scaling H_V converges to $V^{1/3}x_+$, which satisfies Tanner's law.

$$u^\varepsilon(t,x) = \varepsilon h^\varepsilon(\varepsilon^7|\ln\varepsilon|t, \varepsilon x) = \varepsilon H_{\varepsilon^7|\ln\varepsilon|V}(\varepsilon x - \varepsilon^7|\ln\varepsilon|Vt) \to V^{1/3}x_+.$$

Results for the non-homogeneous equation: Growth of the support

Estimates for the growth of the apparent support (Giacomelli-Otto, CPAM 02).

Results for the non-homogeneous equation: Growth of the support

Estimates for the growth of the apparent support (Giacomelli-Otto, CPAM 02).

After the re-scaling:

Results for the non-homogeneous equation: Growth of the support

Estimates for the growth of the apparent support (Giacomelli-Otto, CPAM 02).

After the re-scaling: there exists a $C > 1$ such that

$$C^{-1}\left(t\frac{|\ln\varepsilon|}{\ln t + \ln(\varepsilon^7 \ln\varepsilon)}\right)^{1/7} \le |\{u^\varepsilon > \varepsilon\}| \le C\left(t\frac{|\ln\varepsilon|}{\ln t + \ln(\varepsilon^7 \ln\varepsilon)}\right)^{1/7}.$$

for every $t \in (C, C/\varepsilon^7|\ln\varepsilon|)$.

Results for the non-homogeneous equation: Growth of the support

Estimates for the growth of the apparent support (Giacomelli-Otto, CPAM 02).

After the re-scaling: there exists a $C > 1$ such that

$$C^{-1}\left(t\frac{|\ln\varepsilon|}{\ln t+\ln(\varepsilon^7\ln\varepsilon)}\right)^{1/7}\leq|\{u^\varepsilon>\varepsilon\}|\leq C\left(t\frac{|\ln\varepsilon|}{\ln t+\ln(\varepsilon^7\ln\varepsilon)}\right)^{1/7}.$$

for every $t\in(C, C/\varepsilon^7|\ln\varepsilon|)$.

The result is achieved by using Lagrangian coordinates and energy inequalities.

Results for the non-homogeneous equation: Growth of the support

Estimates for the growth of the apparent support (Giacomelli-Otto, CPAM 02).

After the re-scaling: there exists a $C > 1$ such that

$$C^{-1}\left(t\frac{|\ln\varepsilon|}{\ln t + \ln(\varepsilon^7 \ln\varepsilon)}\right)^{1/7} \le |\{u^\varepsilon > \varepsilon\}| \le C\left(t\frac{|\ln\varepsilon|}{\ln t + \ln(\varepsilon^7 \ln\varepsilon)}\right)^{1/7}.$$

for every $t \in (C, C/\varepsilon^7|\ln\varepsilon|)$.

The result is achieved by using Lagrangian coordinates and energy inequalities.

A notion of apparent support is introduced ($\sim \{u^\varepsilon > \varepsilon\}$).

Our contribution: Key observation

We introduce a new notion of apparent support:

Our contribution: Key observation

We introduce a new notion of apparent support:

$$\rho^\varepsilon = B^\varepsilon(u^\varepsilon),$$

where

$$B^\varepsilon(s) = \frac{1}{|\ln \varepsilon|}\left[\frac{s}{\varepsilon}\arctan\left(\frac{\varepsilon}{s}\right) + \frac{1}{2}\ln\left(1 + \left(\frac{s}{\varepsilon}\right)^2\right)\right].$$

Our contribution: Key observation

We introduce a new notion of apparent support:

$$\rho^\varepsilon = B^\varepsilon(u^\varepsilon),$$

where

$$B^\varepsilon(s) = \frac{1}{|\ln \varepsilon|}\left[\frac{s}{\varepsilon}\arctan\left(\frac{\varepsilon}{s}\right) + \frac{1}{2}\ln\left(1 + \left(\frac{s}{\varepsilon}\right)^2\right)\right].$$

Giacomelli-Otto use Lagrangian coordinates.

Properties of B^ε and ρ^ε

$$\lim_{\varepsilon \to 0} B^\varepsilon(s) = 1 \qquad \text{for any } s > 0,$$

Properties of B^ε and ρ^ε

$$\lim_{\varepsilon\to 0} B^\varepsilon(s) = 1 \qquad \text{for any } s > 0,$$

$$B^{\varepsilon\prime\prime}(s) = -\frac{1}{|\ln\varepsilon|}\frac{1}{s^2+\varepsilon s},$$

Properties of B^ε and ρ^ε

$$\lim_{\varepsilon\to 0} B^\varepsilon(s) = 1 \qquad \text{for any } s > 0,$$

$$B^{\varepsilon\prime\prime}(s) = -\frac{1}{|\ln\varepsilon|}\frac{1}{s^2+\varepsilon s},$$

$$\rho^\varepsilon \sim \begin{cases} 0 & u^\varepsilon < \varepsilon \\ 1-a & u^\varepsilon \sim \varepsilon^{1-a}, \quad a\in(0,1) \\ 1 & u^\varepsilon \geq \frac{1}{|\ln\varepsilon|}. \end{cases}$$

Properties of B^ε and ρ^ε

$$\lim_{\varepsilon\to 0} B^\varepsilon(s) = 1 \qquad \text{for any } s > 0,$$

$$B^{\varepsilon\prime\prime}(s) = -\frac{1}{|\ln\varepsilon|}\frac{1}{s^2+\varepsilon s},$$

$$\rho^\varepsilon \sim \begin{cases} 0 & u^\varepsilon < \varepsilon \\ 1-a & u^\varepsilon \sim \varepsilon^{1-a}, \quad a \in (0,1) \\ 1 & u^\varepsilon \geq \frac{1}{|\ln\varepsilon|}. \end{cases}$$

The choice of B^ε is done so that we have simplification

$$\partial_t \rho^\varepsilon \sim -u^\varepsilon u^\varepsilon_x u^\varepsilon_{xxx} \doteq T^\varepsilon.$$

Ideally

$$\rho^\varepsilon \to \rho \qquad u^\varepsilon \to u$$

Ideally

$$\rho^{\varepsilon} \to \rho \qquad u^{\varepsilon} \to u$$

$$\rho = \chi_{\{u(t)>0\}},$$

Ideally

$$\rho^\varepsilon \to \rho \qquad u^\varepsilon \to u$$

$$\rho = \chi_{\{u(t)>0\}},$$

u is a solution of the Quasi-Static approx with Tanner's law.

or

Ideally

$$\rho^{\varepsilon} \to \rho \qquad u^{\varepsilon} \to u$$

$$\rho = \chi_{\{u(t)>0\}},$$

u is a solution of the Quasi-Static approx with Tanner's law.

or

ρ is the indicator of a solution to the Quasi-Static approx with Tanner's law.

Results

Lemma

$$\{\rho^{\varepsilon}\} \text{ is relatively compact in } L^p(W^{-1,1})$$

Results

Lemma

$$\{\rho^\varepsilon\} \text{ is relatively compact in } L^p(W^{-1,1})$$

Pick $\varepsilon_k \to 0$ such that

$$\rho^{\varepsilon_k} \to \rho.$$

Results

Lemma

$\{\rho^\varepsilon\}$ is relatively compact in $L^p(W^{-1,1})$

Pick $\varepsilon_k \to 0$ such that

$$\rho^{\varepsilon_k} \to \rho.$$

Remark: We can not expect equi-continuity of $\{u^\varepsilon\}_{\{\varepsilon>0\}}$ in any weak norm.

Results

Lemma

$$\{\rho^{\varepsilon}\} \textit{ is relatively compact in } L^p(W^{-1,1})$$

Pick $\varepsilon_k \to 0$ such that

$$\rho^{\varepsilon_k} \to \rho.$$

Remark: We can not expect equi-continuity of $\{u^{\varepsilon}\}_{\{\varepsilon>0\}}$ in any weak norm. Droplets merging instantaneously implies the limit is not continuous. (http://bit.ly/MergedDrops)

Results

Theorem (D., Mellet, to appear CPAM)

$$0 \leq \rho \leq 1 \qquad \& \qquad \partial_t \rho \geq 0.$$

Results

Theorem (D., Mellet, to appear CPAM)

$$0 \leq \rho \leq 1 \qquad \& \qquad \partial_t \rho \geq 0.$$

For a.e. t, every accumulation point w of $\{u^{\varepsilon_k}(t)\}$ satisfies

$$\int_\Omega w = 1, \quad w_{xxx} = 0 \textit{ on } \{w > 0\},$$

Results

Theorem (D., Mellet, to appear CPAM)

$$0 \leq \rho \leq 1 \qquad \& \qquad \partial_t \rho \geq 0.$$

For a.e. t, every accumulation point w of $\{u^{\varepsilon_k}(t)\}$ *satisfies*

$$\int_\Omega w = 1, \quad w_{xxx} = 0 \text{ on } \{w > 0\},$$

$$\{w > 0\} \subset \{\rho = 1\}^o \quad \& \quad \partial\{w > 0\} \subset \partial(\{\rho = 1\}^o).$$

Results

Theorem (D., Mellet, to appear CPAM)

$$0 \leq \rho \leq 1 \qquad \& \qquad \partial_t \rho \geq 0.$$

For a.e. t, every accumulation point w of $\{u^{\varepsilon_k}(t)\}$ *satisfies*

$$\int_\Omega w = 1, \quad w_{xxx} = 0 \text{ on } \{w > 0\},$$

$$\{w > 0\} \subset \{\rho = 1\}^o \quad \& \quad \partial\{w > 0\} \subset \partial(\{\rho = 1\}^o).$$

We can pick a selection of accumulation points $\tilde{w}$, *such that*

$$\partial_t \int_\Omega \rho \geq \int_{\partial\{\tilde{w}>0\}} \frac{|\tilde{w}_x|^3}{3} \, d\mathcal{H}^0 \qquad \text{in } \mathcal{M}_+(0, T).$$

Corollary

Corollary

If $\{\rho = 1\}^o = (a, b)$, *then* u^{ε_k} *converges strongly to*

$$w = 6\frac{(b-x)_+(x-a)_+}{(b-a)^3}.$$

Corollary

Corollary

If $\{\rho = 1\}^o = (a, b)$, *then* u^{ε_k} *converges strongly to*

$$w = 6\frac{(b-x)_+(x-a)_+}{(b-a)^3}.$$

Corollary

$$\partial_t \int_\Omega \rho \geq 144 \left(\int_\Omega \rho\right)^{-6}.$$

Comparisson with literature

Noticing that

$$\int_\Omega \rho^\varepsilon \leq |\{u > \varepsilon\}| + C\frac{|\Omega|}{|\ln \varepsilon|}.$$

Comparisson with literature

Noticing that

$$\int_\Omega \rho^\varepsilon \leq |\{u > \varepsilon\}| + C\frac{|\Omega|}{|\ln \varepsilon|}.$$

We have

$$\lim_{\varepsilon\to 0} |\{u(t) > \varepsilon\}| \geq \int_\Omega \rho(t) \geq (1008\, t + |\{u_{in} > 0\}|^7)^{1/7}.$$

Strategy: A priori estimates

Using the equation

$$\partial_t u^\varepsilon + |\ln \varepsilon|((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx})_x = 0.$$

Strategy: A priori estimates

Using the equation

$$\partial_t u^\varepsilon + |\ln\varepsilon|((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx})_x = 0.$$

Multiplying by u_{xx} and integrating, we have the Energy inequality

$$\int |\partial_x u(T)|^2 + 2|\ln\varepsilon| \int_0^T \int_{\{u>0\}} (u^3 + \varepsilon u^2)|u_{xxx}|^2 \leq \int |\partial_x u_{in}|^2$$

Strategy: A priori estimates

Using the equation

$$\partial_t u^\varepsilon + |\ln\varepsilon|((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx})_x = 0.$$

Multiplying by u_{xx} and integrating, we have the Energy inequality

$$\int |\partial_x u(T)|^2 + 2|\ln\varepsilon| \int_0^T \int_{\{u>0\}} (u^3 + \varepsilon u^2)|u_{xxx}|^2 \leq \int |\partial_x u_{in}|^2$$

For every $t > 0$, u^ε is pre-compact in C^0.

Strategy: A priori estimates

Using the equation

$$\partial_t u^\varepsilon + |\ln\varepsilon|((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx})_x = 0.$$

Multiplying by u_{xx} and integrating, we have the Energy inequality

$$\int |\partial_x u(T)|^2 + 2|\ln\varepsilon| \int_0^T \int_{\{u>0\}} (u^3 + \varepsilon u^2)|u_{xxx}|^2 \leq \int |\partial_x u_{in}|^2$$

For every $t > 0$, u^ε is pre-compact in C^0. Moreover, it converges in $C^2(\{u > 0\})$ to a parabola.

Strategy: A priori estimates

Using the equation

$$\partial_t u^\varepsilon + |\ln \varepsilon|((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx})_x = 0.$$

Multiplying by u_{xx} and integrating, we have the Energy inequality

$$\int |\partial_x u(T)|^2 + 2|\ln \varepsilon| \int_0^T \int_{\{u>0\}} (u^3 + \varepsilon u^2)|u_{xxx}|^2 \leq \int |\partial_x u_{in}|^2$$

For every $t > 0$, u^ε is pre-compact in C^0. Moreover, it converges in $C^2(\{u > 0\})$ to a parabola.

Lemma (Optimal regularity)

$$\|\partial_x u\|_\infty^4 \leq C \left(1 + |\ln \varepsilon| \int_{\{u>0\}} (u^3 + \varepsilon u^2)|u_{xxx}|^2 \right)$$

Strategy: A priori estimates

Using the equation

$$\partial_t u^\varepsilon + |\ln \varepsilon|((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx})_x = 0.$$

Multiplying by $B'_\varepsilon(u^\varepsilon)$ and integrating, we have the Entropy inequality

$$\int \rho^\varepsilon_{in} + 2\int_0^T \int u|u_{xx}|^2 \leq \int \rho^\varepsilon(T)$$

Strategy: A priori estimates

Using the equation

$$\partial_t u^\varepsilon + |\ln \varepsilon|((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx})_x = 0.$$

Multiplying by $B'_\varepsilon(u^\varepsilon)$ and integrating, we have the Entropy inequality

$$\int \rho^\varepsilon_{in} + 2\int_0^T \int u|u_{xx}|^2 \leq \int \rho^\varepsilon(T) \leq 2|\Omega|.$$

Strategy: A priori estimates

Using the equation

$$\partial_t u^\varepsilon + |\ln \varepsilon|((u^{\varepsilon 3} + \varepsilon^{3-n} u^{\varepsilon n}) u^\varepsilon_{xxx})_x = 0.$$

Multiplying by $B'_\varepsilon(u^\varepsilon)$ and integrating, we have the Entropy inequality

$$\int \rho^\varepsilon_{in} + 2\int_0^T \int u|u_{xx}|^2 \leq \int \rho^\varepsilon(T) \leq 2|\Omega|.$$

Observation: B_ε is chosen with this cancellation in mind.

Strategy

Analyze the time derivative of ρ^ε:

$$\partial_t \rho^\varepsilon = \partial_x R^\varepsilon + T^\varepsilon$$

where

$$R^\varepsilon = -|\ln(\varepsilon)| B^{\varepsilon\prime}(u^\varepsilon)(\varepsilon^{3-n} u^{\varepsilon(n-1)} + u^{\varepsilon 2}) u^\varepsilon u^\varepsilon_{xxx}$$

$$T^\varepsilon = -u^\varepsilon u^\varepsilon_x u^\varepsilon_{xxx}.$$

Strategy

Analyze the time derivative of ρ^ε:

$$\partial_t \rho^\varepsilon = \partial_x R^\varepsilon + T^\varepsilon$$

where

$$R^\varepsilon = -|\ln(\varepsilon)| B^{\varepsilon\prime}(u^\varepsilon)(\varepsilon^{3-n} u^{\varepsilon(n-1)} + u^{\varepsilon 2}) u^\varepsilon u^\varepsilon_{xxx}$$

$$T^\varepsilon = -u^\varepsilon u^\varepsilon_x u^\varepsilon_{xxx}.$$

Lemma

$$\lim_{\varepsilon \to 0} \partial_x R^\varepsilon = 0 \qquad \textit{in } L^2(0, T; W^{-1,1}).$$

Strategy

Hence, $\partial_t \rho^\varepsilon \sim -u^\varepsilon u^\varepsilon_x u^\varepsilon_{xxx}$.

Strategy

Hence, $\partial_t \rho^\varepsilon \sim -u^\varepsilon u_x^\varepsilon u_{xxx}^\varepsilon$.Integrating by parts

$$\int \rho^\varepsilon(t)\phi \sim \int_0^t \left(\int u|u_{xx}|^2\phi - \frac{5}{6}\int u_x^3\phi_x - \frac{1}{2}\int u|u_x|^2\phi_{xx} \right) + \int \rho_{in}^\varepsilon \phi.$$

Strategy

Hence, $\partial_t \rho^\varepsilon \sim -u^\varepsilon u_x^\varepsilon u_{xxx}^\varepsilon$.Integrating by parts

$$\int \rho^\varepsilon(t)\phi \sim \int_0^t \left(\int u|u_{xx}|^2\phi - \frac{5}{6}\int u_x^3\phi_x - \frac{1}{2}\int u|u_x|^2\phi_{xx} \right) + \int \rho_{in}^\varepsilon \phi.$$

By a-priori estimates, we can show that if $u^\varepsilon(t) \to w(t)$, then

$$\lim_{\varepsilon\to 0} -\frac{5}{6}\int u_x^3\phi_x - \frac{1}{2}\int u|u_x|^2\phi_{xx} = -\frac{5}{6}\int w_x^3\phi_x - \frac{1}{2}\int w|w_x|^2\phi_{xx}$$

Strategy

Hence, $\partial_t \rho^\varepsilon \sim -u^\varepsilon u^\varepsilon_x u^\varepsilon_{xxx}$.Integrating by parts

$$\int \rho^\varepsilon(t)\phi \sim \int_0^t \left(\int u|u_{xx}|^2\phi - \frac{5}{6}\int u_x^3\phi_x - \frac{1}{2}\int u|u_x|^2\phi_{xx} \right) + \int \rho^\varepsilon_{in}\phi.$$

By a-priori estimates, we can show that if $u^\varepsilon(t) \to w(t)$, then

$$\lim_{\varepsilon\to 0} -\frac{5}{6}\int u_x^3\phi_x - \frac{1}{2}\int u|u_x|^2\phi_{xx} = -\frac{5}{6}\int w_x^3\phi_x - \frac{1}{2}\int w|w_x|^2\phi_{xx}$$

and

$$\liminf_{\varepsilon\to 0}\int u|u_{xx}|^2\phi \geq \int_{\{w>0\}} w|w_{xx}|^2\phi.$$

Strategy

By Fatou's lemma

$$\int \rho(t)\phi \geq \int_0^t \left(\int_{\{w>0\}} w|w_{xx}|^2\phi - \frac{5}{6}\int w_x^3\phi_x - \frac{1}{2}\int w|w_x|^2\phi_{xx} \right) + \int \rho_{in}\phi.$$

Strategy

By Fatou's lemma

$$\int \rho(t)\phi \geq \int_0^t \left(\int_{\{w>0\}} w|w_{xx}|^2\phi - \frac{5}{6}\int w_x^3\phi_x - \frac{1}{2}\int w|w_x|^2\phi_{xx} \right) + \int \rho_{in}\phi.$$

Using that $w_{xxx} = 0$ on $\{w > 0\}$, we have

$$\int \rho(t)\phi \geq \int_0^t \left(\int_{\partial\{w>0\}} \frac{|w_x|^3}{3}\phi \, d\mathcal{H}^0 \right) + \int \rho_{in}\phi$$

Strategy

By Fatou's lemma

$$\int \rho(t)\phi \geq \int_0^t \left(\int_{\{w>0\}} w|w_{xx}|^2\phi - \frac{5}{6}\int w_x^3\phi_x - \frac{1}{2}\int w|w_x|^2\phi_{xx} \right) + \int \rho_{in}\phi.$$

Using that $w_{xxx} = 0$ on $\{w > 0\}$, we have

$$\int \rho(t)\phi \geq \int_0^t \left(\int_{\partial\{w>0\}} \frac{|w_x|^3}{3}\phi \, d\mathcal{H}^0 \right) + \int \rho_{in}\phi$$

This is the inequality form for Tanner's law.

It follows $\partial_t \rho \geq 0$.

Strategy

By Fatou's lemma

$$\int \rho(t)\phi \geq \int_0^t \left(\int_{\{w>0\}} w|w_{xx}|^2\phi - \frac{5}{6}\int w_x^3\phi_x - \frac{1}{2}\int w|w_x|^2\phi_{xx} \right) + \int \rho_{in}\phi.$$

Using that $w_{xxx} = 0$ on $\{w > 0\}$, we have

$$\int \rho(t)\phi \geq \int_0^t \left(\int_{\partial\{w>0\}} \frac{|w_x|^3}{3}\phi \, d\mathcal{H}^0 \right) + \int \rho_{in}\phi$$

This is the inequality form for Tanner's law.

It follows $\partial_t \rho \geq 0$. Also, if $\rho(x) = 1$, then $\partial_t \rho(x) = 0$, which implies $x \notin \partial\{w > 0\}$ and the properties for w follow.

Main Obstacles

- u^{ε} is not even uniformly weakly continuous in time:

Main Obstacles

- u^{ε} is not even uniformly weakly continuous in time:

  Droplets merge at a different time-scale. No easy compactness to identify w.

Main Obstacles

- u^{ε} is not even uniformly weakly continuous in time:

  Droplets merge at a different time-scale. No easy compactness to identify w.
- We can not identify w by its mass on each connected component:

Main Obstacles

- u^{ε} is not even uniformly weakly continuous in time:

  Droplets merge at a different time-scale. No easy compactness to identify w.

- We can not identify w by its mass on each connected component:

  Ostwald Ripening: mass can flow from "disconected" droplets, without merging. (Glasner, Otto, Rump, Slepcev, EJPAM 09)

Main Obstacles

- We can not show

$$\lim_{k\to\infty}\int_0^T\int u^{\varepsilon_k}|u^{\varepsilon_k}_{xx}|^2=\int_0^T\int_{\{w>0\}} w|w_{xx}|^2.$$

Main Obstacles

- We can not show

$$\lim_{k\to\infty}\int_0^T\int u^{\varepsilon_k}|u_{xx}^{\varepsilon_k}|^2=\int_0^T\int_{\{w>0\}}w|w_{xx}|^2.$$

It scales like the Lipschitz norm and formally is slightly weaker than the dissipation:

If

$$u=x_+\log^{1/3}(x),$$

then dissipation is bounded, but

$$\int u|u_{xx}|^2=\infty.$$

Conditional result

If the wetted region is a single interval

$$\{\rho = 1\} = (a(t), b(t)) \qquad \text{for all } t > 0,$$

then can identify w explicitly.

Conditional result

If the wetted region is a single interval

$$\{\rho = 1\} = (a(t), b(t)) \qquad \text{for all } t > 0,$$

then can identify w explicitly. Moreover, if

$$\lim_{k\to\infty} \int_0^T \int u^{\varepsilon_k} |u_{xx}^{\varepsilon_k}|^2 = \int_0^T \int_{\{w>0\}} w|w_{xx}|^2,$$

then ρ satisfies Tanner's law.

That's it for now!

Thank you!