Problem sheet functional analysis

September 24, 2020

1. Considering functional spaces in $\mathbb{R}^n$, a functional semi-norm $|\cdot|_X$ is said to be homogeneous of degree α if for every $\lambda > 0$
$$|f_\lambda|_X = \lambda^\alpha |f|_X,$$
where
$$f_\lambda(x) = f(\lambda x).$$
Find the homogeneity of the following semi-norms:

   - $\|f\|_p = \left(\int_{\mathbb{R}^n} |f|^p \, dx\right)^{\frac{1}{p}}$.
   - $|f|_{1,p} = \left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx\right)^{\frac{1}{p}}$.

   What are the radially symmetric distributions T that are invariant under the transformation the homogeneous transformation
$$T \to \lambda^{-\alpha} T_\lambda?$$
Is it unique? In the cases of the above semi-norm do they have finite semi-norm?

   Remark: For Sobolev spaces α is called the Sobolev number and it is a measure of how regular the functions are in this space.

2. Find the Fourier transform of the tempered distribution

$$f_\beta = |x|^\beta \qquad \beta > -n$$

in terms of the ambient space dimension $\mathbb{R}^n$. How do you have to adapt the f_β with $\beta < -n$, to make it an homogeneous distribution of degree $-d$?

3. For the following decide whether or not T_i is a distribution $T_i : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ and if so determine its order (it is expected that you prove all the claims you make!)

1. $T_1(\varphi) = \sum_{j=1}^{\infty} 2^{-j} \varphi^{(j)}(0)$
2. $T_2(\varphi) = \sum_{j=1}^{\infty} 2^{j} \varphi^{(j)}(j)$
3. $T_3 = P.V.(1/x)$ (Principal Value of $1/x$) defined by

$$T_3(\varphi) = \lim_{\epsilon \to 0^+} \int_{|x|>\epsilon} \frac{\varphi(x)}{|x|}\, dx.$$

4. $T_4 = -\partial_x P.V.(1/x)$.
5. A linear map $L : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ that satisfies $L(\varphi) \geq 0$ for all $\varphi \in C_c^\infty(\mathbb{R}; [0, \infty))$.

4. Let $\Omega \subset \mathbb{R}^n$ be open and bounded

   1. Prove that, if $u \in L^1_{loc}$ is weakly differentiable, then $u_\epsilon := (\epsilon^2 + u^2)^{\frac{1}{2}}$ is also weakly differentiable with
   $$\nabla u_\epsilon = \frac{u}{u_\epsilon} \nabla u.$$
   Conclude that, if $u \in W^{1,p}(\Omega)$, then $u_\epsilon \in W^{1,p}(\Omega)$.
   2. Let $1 \leq p < \infty$. Prove that, if $u \in W^{1,p}(\Omega)$, then $|u| \in W^{1,p}(\Omega)$ and $sign(u)\nabla u$ a.e..

5. Let $\Omega = B_1(0) \subset \mathbb{R}^n$ (unit ball), and let $\alpha > 0$. Show that there exists a constant $C < \infty$ so that

$$\int_\Omega u^2 \, dx \leq C \int_\Omega |\nabla u|^2 \, dx$$

for any $u \in H^1(\Omega)$ with the property that $\mathcal{L}^n(\{x \in \Omega \, : \, u(x) = 0\}) > \alpha$.

6. 1. Let $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^+$. Show that there exists a unique bounded linear operator

$$tr : W^{1,p}(\Omega) \to L^p(\mathbb{R}^{n-1})$$

so that $tr(u) = u|_{x_n=0}$ for $u \in C^1(\overline{\Omega}) \cap W^{1,p}(\Omega)$.

Furthermore, determine the maximal number $q > p$ so that $tr(u) \in L^q_{loc}(\mathbb{R}^{n-1})$ for every $u \in W^{1,p}(\Omega)$. (HINT: Modify the argument for the Sobolev-Embedding theorem)

2. Let $f, g \in W^{1,p}_{loc}(\mathbb{R}^n)$, $1 \leq p < \infty$, and define

$$h(x) := \begin{cases} f(x) & x_n \geq 0 \\ g(x) & x_n < 0 \end{cases},$$

when is $h \in W^{1,p}_{loc}(\mathbb{R}^n)$?

You may use that $C^1(\overline{\Omega}) \cap W^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ is dense for $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^+$.