Mixing and Enhanced Dissipation

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Milk and Coffee video: https://goo.gl/Uc6jJS

Navier-Stokes Equation

In $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$), we consider the equation for an incompressible fluid

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \nu \Delta \boldsymbol{u} + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} = 0. \end{cases} \tag{NSE}$$

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We are interested in the dynamics at high Reynolds number $\nu \ll 1$.

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Question

Given a steady state $(\boldsymbol{u}^S, \omega^S)$, what can we say about its stability?

Linearizing

We write $\omega = \omega^S + \tilde{\omega}$, to first order

$$\begin{cases} \partial_t \tilde{\omega} + \boldsymbol{u}^S \cdot \nabla \tilde{\omega} + \boldsymbol{u} \cdot \nabla \omega^S = \nu \Delta \tilde{\omega}, \\ \boldsymbol{u} = \nabla^{\perp} \tilde{\psi}, \qquad \Delta \tilde{\psi} = \tilde{\omega}. \end{cases}$$

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Then

$$\begin{cases} \partial_t \tilde{\omega} + u\partial_x \omega - u'' \partial_x \tilde{\psi} = \nu \Delta \tilde{\omega}, \\ \tilde{\psi} = \Delta^{-1} \tilde{\omega}. \end{cases}$$

Decay

In abstract terms

$$\partial_t \tilde{\omega} + B\tilde{\omega} + \nu A\tilde{\omega} = 0,$$

$$B = u\partial_x - u''\partial_x\Delta^{-1} \qquad A = -\Delta.$$

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Does $\tilde{w}$ decay to zero? In which sense? If we know decay for $\nu = 0$, under which conditions can we use it for $\nu > 0$?

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- Kolmogorov flow: $u(y) = \sin y$ in $\mathbb{T}^2$. The operator $\sin y \partial_x (1 + \Delta^{-1})$ is anti-symmetric w.r.t.

$$\langle \phi_1, \phi_2 \rangle = \int (1 + \Delta^{-1}) \phi_1 \phi_2 \mathrm{d}x\mathrm{d}y.$$

Mixing estimates

Meta-Theorem

For $\nu = 0$, we have an estimate of the form

$$\|\tilde{w}(t) - P\tilde{w}_{in}\|_{H^{-1}} \lesssim \frac{1}{t^p}\|\tilde{w}_{in} - P\tilde{w}_{in}\|_{H^1}.$$

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Question (Relaxation/Metastability)

How can we use this estimate to show that for $\nu > 0$

$$\|\tilde{w}^{\nu}(t) - e^{t\nu\Delta}P\tilde{w}_{in}\|_{L^2}^2 \lesssim e^{-t\nu^q}\|\tilde{w}_{in} - P\tilde{w}_{in}\|_{L^2}^2,$$

for $q(p) \in (0,1)$ (faster than the heat equation time scale)?

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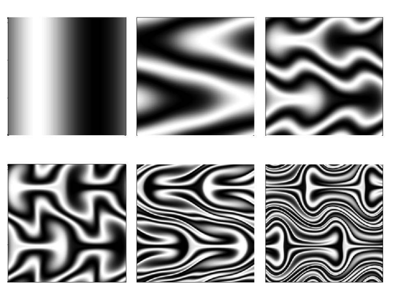

Figure: From E Lunasin, Z Lin, A Novikov, A Mazzucato, C. Doering

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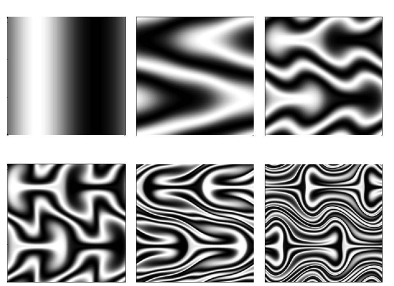

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Smaller and small length scales appear ("cascade")

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Enhanced dissipation

For the viscous problem

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Question

Can we do better?

Enhanced Dissipation

Definition

$\boldsymbol{u}$ is called relaxation enhancing if $\forall\delta > 0$, there exists $\nu_0 = \nu_0(\delta)$ such that for any $\nu < \nu_0$ and any $f^{in} \in L^2$ we have

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Mixing implies $B = \boldsymbol{u}\cdot\nabla$ has no H^1 eigenfunctions.

Some inviscid results

For passive scalar: $\partial_t f + u\partial_x f = 0$, where u has a finite number of critical points, with u' vanishing at order $n_0 \in \mathbb{N}$:

Which	Who	When	H^{-1}	# of pages
$u(y) = y$	Kelvin	1887	$1/t$	$1/2$
$u(y)$	Bedrossian, Coti Zelati	2015	$1/t^{\frac{1}{n_0+1}}$	1

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u mono, u'' small	Zillinger	2015	$1/t$	49
u monotone	Wei, Zhang et al	2015	$1/t$	56+76
$u(y) = \sin y$	Wei, Zhang, et al	2017	$1/t$	92

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A induces the Sobolev space:

$$\|f\|_{H^s} = \|A^{s/2} f\|_H.$$

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We can replace $\boldsymbol{u} \cdot \nabla$ by $B(t)$, such that it is antisymmetric and there exists c_B and $s_0 > 0$ satisfying

$$\|B(t)f\|_H \leq c_B\|f\|_{H^{s_0}} \qquad \& \qquad |Re\langle B(t)f, Af\rangle| \leq c_B\|f\|_{H^1}^2.$$

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We can replace $\boldsymbol{u} \cdot \nabla$ by $B(t)$, such that it is antisymmetric and there exists c_B and $s_0 > 0$ satisfying

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If $B = \boldsymbol{u} \cdot \nabla$ with $\boldsymbol{u}$ divergence free, then

$$\begin{aligned} |\langle B(t)f, Af\rangle| &= \left| \int \boldsymbol{u} \cdot \nabla f (-\Delta) f \,\mathrm{d}x \right| \\ &= \left| \int \nabla \boldsymbol{u}[\nabla f, \nabla f] + \boldsymbol{u} \cdot \nabla \frac{|\nabla f|^2}{2} \,\mathrm{d}x \right| \\ &\leq \|\boldsymbol{u}\|_{L_t^\infty W_x^{1,\infty}} \|f\|_{H^1}^2. \end{aligned}$$

Our contribution: In general

Theorem

Under the previous assumptions. If the inviscid problem $\partial_t f + B(t - t_0)f$ *satisfies*

$$\|f(t)\|_{H^{-1}} \leq \varrho(t)\|f^{in}\|_{H^1} \qquad \text{for any } t_0 > 0,$$

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$$\|f(t)\|_{H^{-1}} \leq \varrho(t)\|f^{in}\|_{H^1} \qquad \text{for any } t_0 > 0,$$

then the viscous problem $\partial_t f^\nu + Bf^\nu + \nu A f^\nu = 0$ satisfies:

- *if $\varrho(t) \sim t^{-p}$, then*

$$\|f^\nu(t)\|_{L^2} \leq \mathrm{e}^{-c_0 \nu^q t}\|f^{in}\|_{L^2}, \qquad q = \frac{2}{2+p}.$$

- *if $\varrho(t) \sim \mathrm{e}^{-t}$, then*

$$\|f^\nu(t)\|_{L^2} \leq \mathrm{e}^{-c_0|\ln\nu|^{-2}t}\|f^{in}\|_{L^2}.$$

New Applications

Spiral Flow ($B = \boldsymbol{u}\partial_x$, $A = -\Delta$):

$$\boldsymbol{u}(r,\theta) = r^{1+\alpha}(-\sin\theta, \cos\theta) \qquad \text{with } \alpha \geq 1.$$

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$$\|f^{\nu}(t)\|_{L^2} \leq \mathrm{e}^{-c_0\nu^q t}\|f^{in}\|_{L^2}, \quad q = \frac{4 - p_\alpha}{4 + p_\alpha} \;\text{ with }\; p = \frac{2}{\max\{\alpha, 2\}}.$$

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Fractional Dissipation ($B = \boldsymbol{u}\partial_x$, $A = (-\Delta)^{\gamma/2}$):

$$\partial_t f^\nu + \boldsymbol{u}\partial_x f^\nu + \nu(-\Delta)^{\gamma/2} f^\nu = 0.$$

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Kolmogorov flow

Kolmogorov Flow ($B = \sin(y)\partial_x(I + \Delta^{-1})$, $A = (-\Delta)$):

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$$\|f^\nu(t)\|_{L^2} \leq \mathrm{e}^{-c_0\nu^{-3/5}t}\|f^{in}\|_{L^2}.$$

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Worse than the time scale $\nu^{-1/2}$ by Ibrahim, Maekawa, Masmoudi or Wei, Zhang, Zhao (2017).

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Yes, for $\mathbb{T}^d$ with $d \geq 2$ there exists a time dependent flow $\boldsymbol{u} \in L^\infty_t W^{1,r}_x$ that mixes arbitrary initial data exponentially based on Baker's map. (Elgindi, Zlatos 2018).

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Using our ideas we can get

$$\|f^\nu(t)\|_{L^2} \leq \mathrm{e}^{-|\ln \nu|^{-2} \nu^{\frac{1}{r-1}} t} \|f^{in}\|_{L^2} .$$

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The geodesic flow acts on the unit tangent bundle of the manifold, hence we are working on dimension 3 or higher.

Triple Linkage

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Yes, Triple Linkage (Hunt MacKay 2003). (Video https://www.youtube.com/watch?v=aVjj6VE-tNg)

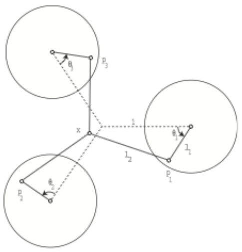
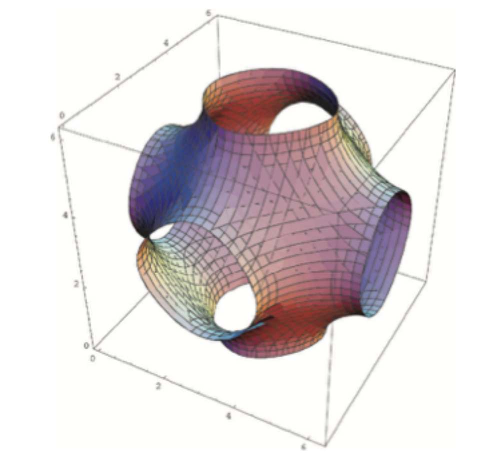


Figure 1. The triple linkage.

Figure: From Hunt MacKay

Triple Linkage



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We have the estimate

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We have just integrated

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Contradiction argument

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$$\nu \|f^\nu(\tau_0)\|_{H^1}^2 < 4\delta\nu^q, \qquad \nu \int_{\tau_0}^{\tau_0+\nu^{-q/2}/2} \|f^\nu(s)\|_{H^1}^2 \mathrm{d}s < 2\delta\nu^{q/2},$$

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We found a time interval where $\|f^\nu(t)\|_{H^1}^2$ is relatively small.

Using the inviscid dynamics

If we tale $f^{\nu}(\tau_0)$ as the initial condition of the inviscid problem, we can estimate that

$$\|f^{\nu}(\tau_0+t)-f(\tau_0+t)\|_{L^2}^2 \leq \frac{1}{4}, \qquad \forall t \in \left[0, \frac{1}{2}\nu^{-q/2}\right].$$

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We do this by looking at the energy accumulated in the high modes.

Inviscid mixing

Also, if $P_{\leq R}$ if the projection onto the span of the eigenfunctions of A corresponding to $|\lambda| \leq R$

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Finally

$$\|f^\nu(\tau_0 + t)\|_{H^1}^2 \geq R\|(I - P_{\leq R}) f^\nu(\tau_0 + t)\|_{L^2}^2 \geq \frac{R}{2}\left(1 - \delta \nu^{q(p+1)-1} R\right).$$

Conclusion

Optimizing over R

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$$2\delta\nu^{q/2} \geq \nu\int_{\frac{1}{4}\nu^{-q/2}}^{\frac{1}{2}\nu^{-q/2}} \|f^\nu(\tau_0+t)\|_{H^1}^2 \mathrm{d}t > \frac{\nu^{-q/2}}{\delta\nu^{q(p+1)-2}}.$$

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By our choice $q(p+2)-2=0$, we get the contradiction

$$\delta^2 > 1/2.$$

Open questions

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Question

Is there Lipschitz regular, exponentially mixing flows in $\mathbb{T}^2$?

That's it for now!

Thank you!