Functional Analysis

Matias G. Delgadino

University of Oxford

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Email: Matias.Delgadino@maths.ox.ac.uk, **Office**: S2.23

Compactness

Sobolev spaces allow to measure regularity in an integral way versus pointwise bounds for C^α.

They are also a great way to obtain compactness:

Theorem

Given a **tight** *sequence in* $\{f_i\}_{i\in\mathbb{N}} \subset L^2$. *If*

$$\sup \|f_i\|_{H^1} < \infty$$

then $\{f_i\}_{n\in\mathbb{N}}$ *admits a strongly convergent subsequence in* L^2.

Compactness

Proof.

Theorem (Riesz-Kolmogorov)

For $1 \leq p < \infty$, *a family* $\{f_i\}_{i\in I}$ *is pre-compact in* $L^2(\mathbb{R}^n)$, *iff,*

- *It is Bounded*

$$\sup_I \|f_i\|_{L^2} < \infty.$$

- *It is equi-continuous:*

$$\sup_I \|f_i(\cdot + h) - f_i(\cdot)\|_{L^2} \to 0.$$

- *It is tight: for every* $\varepsilon > 0$ *there exists a compact set* $K \subset \mathbb{R}^n$ *such that*

$$\sup_I \int_{K^c} |f_i|^2 \leq \varepsilon.$$

Sobolev Spaces

Definition

Given and open set Ω, $k \in \mathbb{N}$, $p \in [1, \infty]$ and $f \in L^1_{loc}(\Omega)$. We say that $f \in W^{k,p}$, if f is k-times weakly differentiable and $D^\alpha f \in L^p(\Omega)$ for every $|\alpha| \leq k$.And we define the norm

$$\|f\|^p_{W^{k,p}} := \sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p}.$$

Sobolev Spaces

Theorem

If boundary of Ω is C^1 and $1 \leq p < \infty$, then $C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

The guiding principle for proofs in Sobolev spaces is to prove it for smooth functions and the extend it by a density argument.

Sobolev Space

We can ask ourselves what happens in 1-d.

Theorem

If $f \in W^{1,p}((0,1))$, then $f \in C^{\alpha}([0,1])$ with $\alpha = 1 - 1/p$.

Remark: We can even define the value of f at the boundary.

Sobolev Spaces

Functions in Sobolev Spaces can be evaluated in measure zero sets.

Theorem (Trace Theorem)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and with C^1 boundary. Then there exists a (unique) bounded linear operator

$$tr : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

so that

$$tr(u) = u|_{\partial\Omega} \qquad \forall u \in C^0(\overline{\Omega}) \cap W^{1,p}(\Omega).$$

Remark: The trace operator is not surjective.Example of the Laplace problem in a Domain.

Sobolev Space

Theorem (Extension theorem)

Let $\Omega \subset \mathbb{R}^n$ *open, bounded and with* C^1 *boundary. Then for every* $p \in [1, \infty]$ *there exists (not unique) a bounded linear operator:*

$$E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$$

so that

$$Ef|_{\Omega} = f$$

Sobolev Spaces

Sketch of Proof.

Two steps prove the theorem for smooth functions and then extend it by continuity/density.

For the first part we can use partition of unity to localize and diffeomorphisms to flatten the boundary locally, reducing ourselves to half balls.

□

Sobolev Spaces

One of the most important subsets of a given Sobolev space is the zero trace class $W_0^{k,p}$:

Theorem

Let Ω be bounded with C^1 boundary. Then $f \in W^{k,p}(\Omega)$ has $tr(f) = 0 \in L^p(\Omega)$, if and only if, f can be approximated by compactly supported smooth functions $C_c^\infty(\Omega)$.

Sobolev Spaces

Trivially we know that

$$W^{1,p}(\Omega) \subset L^p(\Omega).$$

Moreover, in 1-d we saw that

$$W^{1,p}((0,1)) \subset C^\alpha([0,1]) \subset\subset L^\infty((0,1)) \subset L^p((0,1)).$$

In this class we will focus on two things:

- What is the optimal q such that $W^{1,p} \subset L^q$ with a continuous embedding?
- Is the embedding $W^{1,p} \subset L^p$ compact?

Sobolev Spaces

A reasonable way to look at the embedding is to look at scalings or equivalently singularity.

If we are trying to show a bound of the form

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)},$$

then it has to be stable under scalings. i.e. the constant has to be independent of $\lambda > 0$

$$\lambda^{-n/q}\|f\|_{L^q(\mathbb{R}^n)} = \|f_\lambda\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla f_\lambda\|_{L^p(\mathbb{R}^n)} = C\lambda^{1-\frac{n}{p}}\|\nabla f\|_{L^p(\mathbb{R}^n)}$$

where

$$f_\lambda(x) = f(\lambda x)$$

Sobolev Spaces

If the inequality holds for any f

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)},$$

then, we also know that for an specific f and every $\lambda > 0$:

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{\left(1-\frac{n}{p}\right)+\frac{n}{q}}\|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

Either taking $\lambda \to \infty$ or $\lambda \to 0$, we notice that this inequality can hold only if

$$1 - \frac{n}{p} = -\frac{n}{q}$$

or

$$q = p_* = \frac{np}{n-p}$$

Sobolev Spaces

The same analysis follows if we look at the type of acceptable singularities or decay at infinity:

$$|x|^s \chi_{B_1} \in W^{1,p}(\mathbb{R}^n) \qquad \text{iff} \qquad s > 1 - \frac{n}{p}.$$

$$|x|^s \chi_{B_1} \in L^q(\mathbb{R}^n) \qquad \text{iff} \qquad s > -\frac{n}{q}.$$

$$|x|^s \chi_{B_1^c} \in W^{1,p}(\mathbb{R}^n) \qquad \text{iff} \qquad s < 1 - \frac{n}{p}.$$

$$|x|^s \chi_{B_1^c} \in L^q(\mathbb{R}^n) \qquad \text{iff} \qquad s < -\frac{n}{q}.$$

From this relationships we can obtain that we can satisfy the inequality only if $q = p_*$.

Sobolev Spaces

Theorem (Sobolev-Gagliardo-Niremberg)

For any $1 \leq p < n$ *exists* $C(p,n) \in (0,\infty)$, *such that*

$$\|f\|_{L^{p_*}(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)},$$

with

$$p_* = \frac{np}{n-p}.$$

Remark: It is not valid for $p = n$ and $p_* = \infty$. For $n > 1$, $W^{1,n}$ embeds into BMO a non-trivial subset of L^∞.

Example: In 2d,

$$f = (\log(1/r))^{1/3}\chi_{B_1} \in H^1(\mathbb{R}^2).$$

Sobolev Spaces

Corollary

Let $1 \leq p < n$, $p_* = np/(n-p)$, *and* Ω *and open bounded set with* C^1 *boundary. Then, for every* $1 \leq q \leq p_*$, *there exists* $C(p, q, n)$ *such that*

$$\|f\|_{L^q(\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)}.$$

Sobolev Space

Proof of Corollary.

We require a C^1 boundary to be able to use the extension:

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

$$\|f\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p_*}}\|f\|_{L^{p_*}(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p_*}}\|Ef\|_{L^{p_*}(\mathbb{R}^n)}$$
$$\leq |\Omega|^{\frac{1}{q}-\frac{1}{p_*}} C_{SGN}\|\nabla Ef\|_{L^p(\mathbb{R}^n)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p_*}} C_{SGN} C_E \|f\|_{W^{1,p}(\Omega)}$$

□

Sobolev Spaces

Proof of SGN inequality.

Same philosophy as always, prove it for smooth functions and extend it by density.

We need to show that there exists $C(p, n)$ such that

$$\|f\|_{L^{p*}} \leq C\|\nabla f\|_{L^p} \qquad \forall f \in C_c^\infty(\mathbb{R}^n)$$

Most tractable case, $n = 2$ and $p = 1$:

$$|u(x_1, x_2)| \leq \int_{\mathbb{R}} |\partial_1 u(t, x_2)| \, dt \qquad \text{or} \qquad \int_{\mathbb{R}} |\partial_2 u(x_1, t)| \, dt$$

$\square$

Sobolev Spaces

Proof of SGN inequality.

Multiplying the inequalities:

$$|u(x_1,x_2)|^2 \le \left(\int_{\mathbb{R}} |\partial_1 u(t,x_2)|\, dt\right)\left(\int_{\mathbb{R}} |\partial_2 u(x_1,t)|\, dt\right).$$

Integrating

$$\int_{\mathbb{R}^2} |u(x_1,x_2)|^2\, dx_1 dx_2 \le \left(\int_{\mathbb{R}^2} |\partial_1 u(t,x_2)|\, dt dx_2\right)\left(\int_{\mathbb{R}^2} |\partial_2 u(x_1,t)|\, dt dx_1\right).$$

$$\|u\|^2_{L^2}(\mathbb{R}^2) \le \|\nabla u\|^2_{L^1}(\mathbb{R}^2).$$

$\square$

Sobolev Spaces

Proof of SGN inequality.

For $p = 1$ and $n = 3$

$$|u(x_1, x_2, x_3)|^{\frac{3}{2}} \leq \left(f_1(x_2, x_3) f_2(x_1, x_3) f_3(x_1, x_2)\right)^{1/2}$$

where

$$f_i = \left(\int_{\mathbb{R}} |\partial_i u(t, \hat{x}_i)| \, dt \right)$$

Integrating, in the first variable

$$\int |u(x_1, x_2, x_3)|^{\frac{3}{2}} \, dx_1 \leq f_1^{1/2}(x_2, x_3) \int \left(f_2(x_1, x_3) f_3(x_1, x_2)\right)^{1/2} \, dx_1$$

□

Sobolev Spaces

Proof of SGN inequality.

Integrating, in the first variable and doing Cauchy-Schwarz

$$\int |u(x_1, x_2, x_3)|^{\frac{3}{2}}\ dx_1$$

$$\leq f_1^{1/2}(x_2, x_3) \left(\int f_2(x_1, x_3)\ dx_1 \right)^{1/2} \left(\int f_3(x_1, x_2)\ dx_1 \right)^{1/2}$$

Repeating the process, we obtain the inequality

$$\int_{\mathbb{R}^3} |u|^{\frac{3}{2}}\ dx \leq \|\nabla u\|_{L^1(\mathbb{R}^3)}^{\frac{3}{2}}$$

$\square$

Sobolev Spaces

Proof of SGN inequality.

A similar argument yields

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, dx \leq \|\nabla u\|_{L^1(\mathbb{R}^n)}^{\frac{n}{n-1}}.$$

Remark: This is not the optimal constant! It is given by the isoperimetric inequality

$$|\Omega|^{\frac{n-1}{n}} \leq \frac{1}{n\omega_n^{1/n}} \mathcal{H}^{n-1}(\partial\Omega).$$

$\square$

Sobolev Spaces

For the general SGN inequality consider u^γ, such that

$$\gamma \frac{n}{n-1} = p_* = \frac{np}{n-p}.$$

By the case $p = 1$,

$$\left(\int u^{p_*} \right)^{\frac{n-1}{n}} \leq \int |\nabla (u^\gamma)| = \int |u|^{\gamma-1} |\nabla u|$$

$$\leq \left(\int |u|^{q(\gamma-1)} \right)^{\frac{1}{q}} \|\nabla u\|_{L^p},$$

where $1/p + 1/q = 1$.

Magic happens $q(\gamma - 1) = p_*$

Sobolev Spaces

Corollary

Let Ω be bounded, there exists $C(\Omega)$ such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W_0^{1,p}(\Omega).$$

Remark: In particular, up to constant we can change the norm in $W_0^{1,p}(\Omega)$ to be

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}.$$

Sobolev Spaces

Proof of Corollary.

We know that $C_c^\infty(\Omega)$ dense in $W_0^{1,p}(\Omega)$, then for any $u \in C_c^\infty(\Omega)$ we can trivially extend by zero, i.e. $u \in C_c^\infty(\mathbb{R}^n)$.

$$\|u\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p}-\frac{1}{p*}} \|u\|_{L^{p*}(\Omega)} \leq |\Omega|^{\frac{1}{p}-\frac{1}{p*}} \|u\|_{L^{p*}(\mathbb{R}^n)}$$
$$\leq C_{SGN}|\Omega|^{\frac{1}{p}-\frac{1}{p*}} \|\nabla u\|_{L^p}.$$

$\square$

Sobolev Spaces

Theorem (Rellich-Kondrachov)

Let Ω be a bounded set with $\partial\Omega$ C^1, and $1 \leq p < n$. Then, the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for any $1 \leq q < p_*$. i.e. given a sequence $\{u_n\} \subset W^{1,p}(\Omega)$ if

$$\sup \|u_n\|_{W^{1,p}(\Omega)} < \infty,$$

then it admits a strongly convergent subsequence in $L^q(\Omega)$.

Remark: It is not valid for $q = p_*$, but you can characterize the behavior exactly.

Proof.

From the Riesz-Kolmogorov theorem (we have used an extension), we need to check boundedness, equi-continuity and tightness.Boundedness and tightness, follow from the bound on the $W^{1,p}$ norm and compactness of Ω, respectively.

For $q = p$, we have

$$\int_\Omega |u(x+h) - u(x)|^p \, dx \le \int_\Omega \int_0^1 |\partial_t u(x+th)|^p \, dx \le h^p \|\nabla u\|_{L^p}^p.$$

Equi-continuity in L^p follows from a uniform bound on $W^{1,p}(\Omega)$.

For $q < p$ we need to add an extra Hölder inequality and use the domain is bounded.

For $p < q < p_*$ we need to use the interpolation inequality

$$\|u_h - u\|_{L^q} \le \|u_h - u\|_{L^p}^{1-\lambda} \|u_h - u\|_{L^{p_*}}^{\lambda} \le h^{1-\lambda} \|\nabla u\|_{L^p}.$$

Compactness can be used to derive the following inequality

Theorem (Poincare Inequality)

Let Ω be a bounded set with C^1 boundary and connected. Then exists $C(\Omega, p)$ such that

$$\|u - \overline{u}_\Omega\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W^{1,p}(\Omega).$$

where

$$\overline{u}_\Omega = \frac{1}{|\Omega|}\int_\Omega u \, dx.$$

Corollary

Let $1 \leq p < \infty$. Then exists $C(n,p)$ such that

$$\|u - \overline{u}_{B_r(x_0)}\|_{L^p(B_r(x_0))} \leq Cr\|\nabla u\|_{L^p(B_r(x_0))}.$$

Proof Corollary.

Use the constant for $\Omega = B_1(0)$, then scale and translate. $\square$

Proof of Poincare's Inequality.

The idea is to use compactness, similar to the proof that all norms in $\mathbb{R}^n$ are equivalent.
WLOG, we reduce ourselves to average zero functions and we want to find

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^p}.$$

We can characterize

$$C^{-1} = \inf_{\{u:\|u\|_{L^p}=1\}} \|\nabla u\|_{L^p}.$$

By Rellich-Kondrachov, the minimizing sequence admits a limit that belongs to the set $\{\|u\|_{L^p} = 1\}$, hence it can not be trivial. $\square$

Definition

Let Ω be open and $\alpha \in (0,1]$.

$$C^{0,\alpha}(\Omega) = \{u \in C^0(\Omega) : \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty\}$$

is a Banach space with norm

$$\|u\|_{C^{0,\alpha}} = \|u\|_{L^\infty} + [u]_{C^{0,\alpha}}$$

Theorem

Given Ω with a C^1 boundary and $u \in L^p(\Omega)$, then $u \in C^{0,\alpha}(\Omega)$ if and only if

$$[u]^{int}_{p,p+n\alpha} = \sup_{x\in\Omega,\ r>0} \left(r^{-(n+p\alpha)} \int_{\Omega\cap B_r(x_0)} |u - \overline{u}_{x_0,r}|^p \right)^{1/p} < \infty.$$

This implies that $[u]^{int}_{p,p+n\alpha} \sim [u]_{C^{0,\alpha}}$.

As a consequence we obtain

Theorem (Morrey's Enbedding Theorem)

Let Ω be bounded with a C^1 boundary and $n < p \leq \infty$. Then

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha_*}(\Omega)$$

with $0 < \alpha^* = 1 - n/p \leq 1$.

Remark: Using that $C^{0,\alpha_*}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ is compact, we obtain compactness.

Morrey's embedding Theorem.

We want to estimate

$$|u(x) - u(y)| \leq C|x-y|^{1-n/p}\|\nabla u\|_{L^p}.$$

We consider a ball B of radius $|x-y|$ that contains both

$$|u(x) - u(y)| \leq |u(x) - \overline{u}_B| + |u(y) - \overline{u}_B|$$

We need to show that for any $x \in B$ with radius r_0

$$|u(x) - \overline{u}_B| \leq C r_0^{1-n/p}\|\nabla u\|_{L^p}$$

$\square$

Morrey's embedding Theorem.

$$|u(x) - \overline{u}_B| \leq \left| u(x) - \frac{1}{|B|} \int_B u \right| \leq \frac{1}{|B|} \int_B |u(x) - u(z)| \, dz$$
$$\leq \frac{2^n}{|B_{2r_0}|} \int_{B_{2r_0}(x)} |u(x) - u(z)| \, dz$$

$\square$

Morrey's embedding Theorem.

Using polar coordinates $z = t\omega$ then

$$|u(x) - u(z)| \leq \int_0^t |\nabla u(x_0 + s\omega)| \, ds \leq \int_0^{2r_0} |\nabla u(x_0 + s\omega)| \, ds$$

then

$$\frac{2^n}{|B_{2r_0}|} \int_{B_{2r_0}(x)} |u(x) - u(z)| \, dz \leq \frac{2^n}{|B_{2r_0}|} \int_0^{2r_0} \int_{\mathbb{S}} \int_0^{2r_0} |\nabla u(x_0 + s\omega)| \, ds d\omega dt.$$

Applying Fubini

$$\frac{2^n}{|B_{2r_0}|} \int_{B_{2r_0}(x)} |u(x) - u(z)| \, dz \leq \frac{2^{n+1} r_0}{|B_{2r_0}|} \int_{B_{2r_0}} |\nabla u| \, dx$$

$\square$

Morrey's embedding Theorem.

Finally, applying Hölder

$$\frac{2^{n+1} r_0}{|B_{2r_0}|} \int_{B_{2r_0}} |\nabla u| \, dx \leq C_n \frac{2^{n+1} r_0}{|B_{2r_0}|} |B_{2r_0}|^{1-1/p} \|\nabla u\|_{L^p} = C_n r^{1-n/p} \|\nabla u\|_{L^p}.$$

□