Functional Analysis

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The space of test functions

Given $\Omega \subset \mathbb{R}^n$, we consider the topological vector space $C_c^\infty(\Omega)$, with the following notion of convergence:
Given $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$, we say that

$$\varphi_n \to^{\mathcal{D}} \varphi,$$

if

- All the supports of φ_n are contained in the same compact set

$$\bigcup supp\varphi_n \subset K \subset \Omega$$

- φ_n and all its derivatives converge uniformly to φ:

$$\|D^\alpha \varphi_n - D^\alpha \varphi\|_\infty \to 0 \qquad \forall \alpha.$$

Distributions

Definition

$T : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a distribution if

- T is linear.
- For every $K \subset\subset \Omega$, exists $C(K) > 0$ and $N(k) \in \mathbb{N}$ such that

$$|T(\varphi)| \leq C \sup_{|\alpha| \leq N} \|D^\alpha \varphi\|_\infty \qquad \forall \varphi \in C_c^\infty(K).$$

Remark: We say that a distribution T is of order N_0 if the same N_0 serves for all compact sets: $T \in \mathcal{D}'_{N_0}(\Omega)$.

Distributions

Why do we care about distributions:

- The space C_c^∞ is one of the tiniest spaces we can consider, hence it's dual is huge.
- In the same spirit as measures, they have good compactness properties.
- We can define the dual of all the continuous linear mappings from $\mathcal{D}$ into itself. Like derivatives, convolution (mollificaiton) and Fourier transform.

Distributions

Definition (Distributional Derivative)

Given $T \in \mathcal{D}'$, we define $D^\alpha T \in \mathcal{D}'$ by

$$D^\alpha T(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi).$$

Remark: Derivatives commute, because they commute for C^∞ functions.

Distributions

Examples:

- The derivative of the Heavyside function is the delta.
- We can take finite derivatives of a delta.

Tempered Distributions

There is a highlighted set within distributions, which is called **tempered distributions** denoted by $\mathcal{S}'(\mathbb{R}^n)$ which is the dual of the Schwartz class $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.
This is the class of rapidly decreasing functions

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty : \sup_x ||x|^k D^\alpha f(x)| < \infty \forall \alpha \;\&\; k \in \mathbb{N}\}$$

Tempered Distributions

The main point to define this class is that it is preserved under the Fourier transform:

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

$$\mathcal{F}(\varphi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \varphi(x)\, dx$$

Tempered Distributions

By duality we can extend it to $\mathcal{S}' \subset \mathcal{D}'$.

By mollification arguments we notice that it coincides with the traditional version, when the Fourier transform is classically defined, e.g. $T \in L^2$

Fourier transform

Properties:

- Given a tempered distribution $T \in \mathcal{S}'$

$$\mathcal{F}(\mathcal{F}(T)) = T_-.$$

  Moreover, it is invertible

$$\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(T)))) = T.$$

- Plancherel, given function $f, g \in L^1_{loc}$

$$\int_{\mathbb{R}^n} \mathcal{F}(f)\mathcal{F}(g)\, d\xi = \int_{\mathbb{R}^n} fg\, dx$$

  Therefore,

$$\|\mathcal{F}(f)\|_2 = \|f\|_2$$

Fourier Transform

- Derivatives become multiplication and multiplication becomes derivatives
$$\mathcal{F}(D^\alpha T) = C_\alpha \xi^\alpha \mathcal{F}(T).$$
and
$$\mathcal{F}(x^\alpha T) = C_\alpha^{-1} D^\alpha \mathcal{F}(T).$$

Theorem

Every spatially homogeneous linear PDE has a distributional solution.

Fourier transform

We can also use this to measure regularity:

Definition

We say that a function $f \in L^1_{loc}(\mathbb{R}^n)$ belongs to the Sobolev space $H^1(\mathbb{R}^n)$, if and only if,

$$\|f\|_{H^1}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)|\mathcal{F}(f)|^2 \, d\xi < \infty.$$

By Plancherel's and property of derivatives, this equivalent to

$$f \in L^2(\mathbb{R}^n) \qquad \& \qquad \nabla f \in L^2(\mathbb{R}^n),$$

where ∇f stands for the distributional gradient.

Compactness

Sobolev spaces allow to measure regularity in an integral way versus pointwise bounds for C^α.

They are also a great way to obtain compactness:

Theorem

Given a **tight** *sequence in* $\{f_i\}_{i\in\mathbb{N}} \subset L^2$*. If*

$$\sup \|f_i\|_{H^1} < \infty$$

then $\{f_i\}_{n\in\mathbb{N}}$ *admits a strongly convergent subsequence in* L^2*.*

Compactness

Proof.

Theorem (Riesz-Kolmogorov)

For $1 \leq p < \infty$, a family $\{f_i\}_{i \in I}$ is pre-compact in $L^2(\mathbb{R}^n)$, iff,

- *It is Bounded*
$$\sup_I \|f_i\|_{L^2} < \infty.$$
- *It is equi-continuous:*
$$\sup_I \|f_i(\cdot + h) - f_i(\cdot)\|_{L^2} \to 0.$$
- *It is tight: for every $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that*
$$\sup_I \int_{K^c} |f_i|^2 \leq \varepsilon.$$

Compactness

As we are assuming **tightness**, we need to show **boundedness** and **equicontinuity**.
Boundedness follows from the following inequality and assumption

$$\sup \|f_i\|_{L^2} \le \sup \|f_i\|_{H^1} < \infty$$

Checking **equicontinuity**, we need to look at

$$\|f_i(\cdot + h) - f_i(\cdot)\|_{L^2}^2 = \int \left| e^{-i2\pi h\xi} - 1 \right|^2 |\mathcal{F}(f_i)|^2 \, d\xi,$$

where we have used the formula for the Fourier transofom of a transaltion and Plancherel's.

Compactness

So we need to show that

$$\lim_{h\to 0}\int \left|e^{-i2\pi h\xi}-1\right|^2 |\mathcal{F}(f_i)|^2 \, d\xi \to 0$$

uniformly in $i \in \mathbb{N}$.

We separate the integral into the close field and far field

$$\int \left|e^{-i2\pi h\xi}-1\right|^2 |\mathcal{F}(f_i)|^2 \, d\xi =$$

$$\int_{|\xi|<L} \left|e^{-i2\pi h\xi}-1\right|^2 |\mathcal{F}(f_i)|^2 \, d\xi + \int_{|\xi|>L} \left|e^{-i2\pi h\xi}-1\right|^2 |\mathcal{F}(f_i)|^2 \, d\xi$$

Compactness

For the first integral we use

$$\left|e^{-i2\pi h\xi} - 1\right|^2 \lesssim h^2|\xi|^2$$

and for the second integral

$$\left|e^{-i2\pi h\xi} - 1\right|^2 \leq 4.$$

and

$$\int_{|\xi|>L} |\mathcal{F}(f_i)|^2 \, d\xi \leq \frac{1}{L^2} \int_{|\xi|>L} |\xi|^2 |\mathcal{F}(f_i)|^2 \, d\xi \leq \frac{1}{L^2} \leq \frac{\|f_i\|_{H^1}^2}{L^2}.$$

Compactness

Putting all the inequalities together we obtain

$$\|f_i(\cdot + h) - f_i(\cdot)\|_{L^2}^2 \lesssim \left(L^2 h^2 + \frac{1}{L^2}\right) \|f_i\|_{H^1}^2,$$

which is enough to show the desired equi-continuity.

Remark: We only need tightness of the Fourier transform to make the proof work. In particular, we do not need a full derivative in L^2 by any positive $\alpha > 0$ is enough, i.e. bounded sets of H^α are pre-compact up to tightness.

Sobolev Spaces

We begin with the general definition:

Definition

A function $f \in L^1_{loc}(\Omega)$ is sais to be weakly differentiable in the direction x_i, if there exists $g \in L^1_{loc}(\Omega)$ such that

$$-\int_\Omega f \partial_1 \varphi \, dx = \int_\Omega g \varphi \, dx \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

We say that g is the weak derivative of f.

Remark: Weak derivatives coincide with strong derivatives when $f \in C^1(\Omega)$.In general, we denote ∇f to be the weak/distributional gradient of f, even when this is only a distribution.

We can also define higher order weak derivatives.

Sobolev Spaces

Definition

Given and open set Ω, $k \in \mathbb{N}$, $p \in [1, \infty]$ and $f \in L^1_{loc}(\Omega)$. We say that $f \in W^{k,p}$, if f is k-times weakly differentiable and $D^\alpha f \in L^p(\Omega)$ for every $|\alpha| \leq k$.And we define the norm

$$\|f\|^p_{W^{k,p}} := \sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p}.$$

Sobolev spaces

Variants:

We can take the homogeneous space, defined by it's norm

$$\|f\|^p_{\mathring{W}^{k,p}} := \sum_{|\alpha|=k} \|D^\alpha f\|^p_{L^p}.$$

Locally integrable:

$$\|f\|^p_{W^{k,p}_{loc}} := \sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p_{loc}}.$$

Similar to L^1, it is better if we replace the condition that the weak derivative is in $L^1(\Omega)$ by $\mathcal{M}(\Omega)$

$$W^{1,1}(\Omega) \subset BV(\Omega) = \{f \in L^1 \ : \ \nabla f \in \mathcal{M}(\Omega)\}.$$

Sobolev spaces

Notation: For $p = 2$, we denote $W^{k,2} = H^k$ and for $\Omega = \mathbb{R}^n$ it is equivalent to the Fourier transform definition.

Theorem

If Ω has a smooth boundary or it is convex, then

$$\|f\|_{W^{1,\infty}} \sim \|f\|_{Lip}.$$

Remark: Pacman domain example that it doesn't work in general.

Sobolev Spaces

Remark: For bounded domains $L^p(\Omega) \subset L^q(\Omega)$ if $p \leq q$, which implies

$$W^{k,p}(\Omega) \subset W^{k,q}(\Omega).$$

Remark: We can try to understand how regular functions in $W^{k,p}$ are by looking at radial singularities:

$$|x|^s \in W^{k,p}_{loc} \qquad \text{if and only if} \qquad s > n - \frac{k}{p}.$$

Sobolev Spaces

Remark: χ_{B_1} does not belong to $W^{1,p}$ for any p, but belongs to BV. Similar for any other smooth domain.

Remark: If $f, g \in W^{1,p}$, then $\max\{f, g\} \in W^{1,p}$.Moreover, $|f| \in W^{1,p}$ and

$$\nabla|f| = \frac{f}{|f|}\nabla f \in L^p.$$

Sobolev Space

Theorem

For any open set $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}$ and $p \in [0, \infty]$, the space $W^{k,p}(\Omega)$ is a Banach space.

- *If $1 \leq p < \infty$ it is separable.*
- *If $1 < p < \infty$ it reflexible.*
- *If $p = 2$ it is a Hilbert space.*

Proof.

Show that $W^{k,p}$ is a closed subspace of L^p. □

Sobolev Space

We can ask ourselves what happens in 1-d.

Theorem

$f \in W^{1,p}((0,1))$ if only if f is p Absolutely Continuous (A.C.(p)), i.e. exists $g \in L^p((0,1))$ such that

$$|f(t) - f(s)| \leq \int_t^s g(u)\, du.$$

Remark: $W^{1,p}((0,1)) \subset C^0([0,1])$, note that we can even define it's value up to the boundary.

Sobolev Spaces

More classical definition:

Definition

$f \in C^0((0,1))$ is A.C. (1), if and only if, for every $\varepsilon > 0$ exists $\delta > 0$ such that for every disjoint $\{(a_i, b_i)\}_{i=1}^m$ such that $\sum |b_i - a_i| < \delta$ implies

$$\sum |f(b_i) - f(a_i)| < \varepsilon.$$

Sobolev Spaces

Higher dimensional Sobolev spaces have similar (weaker) properties

Theorem

Let Ω be open and bounded, and $p \in [1, \infty]$. Let $f \in L^p(\Omega)$, then $f \in W^{1,p}(\Omega)$, if and only if, f has a representative such that for each $j \in 1, ..., n$ the function

$$t \to f_j(x', t) = f(x_1, ..., x_{j-1}, t, x_{j+1}, ..., x_n)$$

is A.C. for $\mathcal{L}^{n-1}$-a.e. for $(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) \in \mathbb{R}^{n-1}$, where defined and

$$\partial_t f_t \in L^p(\Omega).$$

Remark: For example $f = x/|x|$ is $C^\infty(\mathbb{R}^n \setminus \{0\})$.

Sobolev Spaces

We know that smooth functions are dense.

Theorem (Meyers-Serrin '64)

Let $1 \leq p < \infty$, then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

Remark: If Ω is not bounded then C^∞ is not contained in $W^{k,p}$.

Sobolev Spaces

Remark: Pacman domain shows that we can't approximate with continuity up to the boundary for general domains.

Theorem

If boundary of Ω *is* C^1 *and* $1 \leq p < \infty$, *then* $C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ *is dense in* $W^{k,p}(\Omega)$.

Proof.

The idea of the proof is to use a partition of unity to localize (and flatten) and then use mollification. □

The guiding principle for proofs in Sobolev spaces is to prove it for smooth functions and the extend it by a density argument.

Sobolev Spaces

Functions in Sobolev Spaces can be evaluated in measure zero sets.

Theorem (Trace Theorem)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and with C^1 boundary. Then there exists a (unique) bounded linear operator

$$tr : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

so that

$$tr(u) = u|_{\partial\Omega} \qquad \forall u \in C^0(\overline{\Omega}) \cap W^{1,p}(\Omega).$$

Notation: We call $tr(u)$ the trace of u onto the boundary.

Remark: The trace operator is not surjective.Example of the Laplace problem in a Domain.

Sobolev Space

Remark: The trace operator can not be extended in a continuous way to $L^p(\Omega)$.
The lack of a trace in L^p can be seen by the following. Given sets $\Omega \subset \tilde{\Omega}$ and $f \in L^p(\Omega)$ it can always be extended by zero on $\tilde{\Omega} \setminus \Omega$ to get $\tilde{f} \in L^p(\tilde{\Omega})$ that coincides with f in Ω. For $f \in W^{1,p}$ this is not good enough!

Theorem (Extension theorem)

Let $\Omega \subset \mathbb{R}^n$ open, bounded and with C^1 boundary. Then for every $p \in [1, \infty]$ there exists (not unique) a bounded linear operator:

$$E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$$

so that

$$Ef|_{\Omega} = f$$

Sobolev Spaces

Sketch of Proof.

Two steps prove the theorem for smooth functions and then extend it by continuity/density.

For the first part we can use partition of unity to localize and diffeomorphisms to flatten the boundary locally, reducing ourselves to half balls.

For trace theorem we need

$$\int_{B_1\cap\{x_n=0\}} |u|^p \, dx' \leq C\|u\|_{W^{1,p}(B_1^+)} \qquad \forall u \in C^\infty(\overline{B_1^+}).$$

□

Sobolev Spaces

For extension theorem with $k > 1$ we need build an extension

$$E : C^\infty(\overline{B_1^+}) \to C^\infty(B_1)$$

satisfying

$$\|Eu\|_{W^{k,p}(B_1)} \leq C\|u\|_{W^{k,p}(B_1^+)}$$

e.g.

$$Eu(x) = \begin{cases} u(x) & x \in \overline{B_1^+} \\ 4u(x', -x_n/2) - 3u(x', -x_n) & x \notin \overline{B_1^+} \end{cases}$$

or higher order approximations.

Sobolev Spaces

One of the most important subsets of a given Sobolev space is the zero trace class $W_0^{k,p}$:

Theorem

Let Ω be bounded with C^1 boundary. Then $f \in W^{k,p}(\Omega)$ has $tr(f) = 0 \in L^p(\Omega)$, if and only if, f can be approximated by compactly supported smooth functions $C_c^\infty(\Omega)$.

Remark: So we can define $W_0^{k,p}$ in two different ways.

Remark: For $\Omega = \mathbb{R}^n$, $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$.

Sobolev Spaces

Based on the duality of L^p with L^q with dual exponents $(1/p + 1/q = 1)$ we can characterize the dual of Sobolev spaces $W^{k,p}$:

Theorem

Let $1 \leq p < \infty$. *Then* $L \in (W_0^{k,p})^*(\Omega)$, *if and only if, there exists* $\{g_\alpha\}_{|\alpha| \leq k} \subset L^q(\Omega)$ *such that*

$$L(f) = \sum_{\alpha \leq k} \int g_\alpha D^\alpha f \, dx.$$

Notation:

$$(W_0^{k,p})^*(\Omega) = W^{-k,q}(\Omega)$$

and

$$(H_0^1)^* = H^{-1}.$$

Sobolev Spaces

Remark: The equivalence of the theorem is an isometric ishomorphism

$$\|L\|_{(W_0^{k,p})^*(\Omega)} = \inf_{g_\alpha} \left(\sum_{\alpha \le k} \|g_\alpha\|_q^q \right)^{1/q} .$$

Sobolev Spaces

To by pass regularity issues, it is usually good to work with difference quotients

$$\partial_i^h f(x) = \frac{f(x + e_i h) - f(x)}{h} \qquad \& \qquad \nabla^h f = (\partial_1^h f, ..., \partial_n^h).$$

We can bound quotients by derivatives

Theorem

Let $1 \leq p < \infty$, and $\Omega' \subset\subset \Omega$, exists $C > 0$ such that

$$\|\nabla^h f\|_{L^p(\Omega')} \leq C\|\nabla f\|_{L^p(\Omega)} \qquad \forall f \in W^{1,p}, \ \& \ h \in (0, dist(\partial\Omega, \Omega')).$$

Proof.

By approximation and density. □

Sobolev Spaces

Conversely, we can bound derivatives by quotients:

Theorem

Let $1 < p < \infty$. Let $f \in L^p_{loc}(\Omega)$ be such that there exists $\Omega' \subset\subset \Omega$ satisfying

$$\sup_{0<h<\varepsilon} \|\nabla^h f\|_{L^p(\Omega')} \infty$$

then $f \in W^{1,p}(\Omega')$ with

$$\|\nabla f\|_{L^p(\Omega')} \leq \sup \|\nabla^h f\|_{L^p(\Omega')}.$$

Proof.

Using Banach-Alaoglu compactness and the characterization of weak derivative. **Remark:** Not valid for $p = 1$. □

Sobolev Spaces

We can apply this to show H^2 regularity to solutions of

$$\begin{cases} \Delta u = f \in L^2(\Omega) \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We can't test with $D^2 u$ because it is not a-priori a function, but we can always test with $(\partial_i^h)^2 u\varphi$ to show

$$\sup_h \int |(\partial_i^h)\nabla u|^2 \varphi < \infty$$

for any φ smooth bump function.