Functional Analysis

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Riesz-Representation Theorem

Other version of the same theorem from last class:

Theorem (Riesz-Representation Theorem L^2)

The dual of a Hilbert space can be identified with the same space

$$H \sim H^*.$$

i.e. for every $T \in H^*$ (i.e. $T : H \to \mathbb{R}$) there exists a unique $g \in H$ such that

$$T(f) = \langle f, T \rangle_{H,H^*} = \int fg \, dx = \langle f, g \rangle_{H,H}.$$

Riesz-Representation Theorem

Yet another version of the same theorem:

Theorem (Riesz-Representation Theorem L^p)

For $1 \leq p < \infty$, the dual of L^p can be identified with L^q i.e. for every $T \in (L^p)^*$ (i.e. $T : L^p \to \mathbb{R}$) there exists a unique $g \in L^q$ such that

$$T(f) = \langle f, T \rangle_{L^p,(L^p)^*} = \int fg \, dx = \langle f, g \rangle_{L^p,L^q}.$$

What happens to L^1

Theorem (Riesz-Representation Theorem)

The dual of continuous functions with compact support is locally finite measures:

$$\mathcal{M}_{loc}(\Omega) = (C_c(\Omega))^*.$$

i.e. for every $T \in (C_c(\Omega))^*$ there exists a unique $\mu \in \mathcal{M}_{loc}(\Omega)$ such that

$$\langle f, T \rangle_{C_c(\Omega),(C_c(\Omega))^*} = \int_{suppf} f \; d\mu = \int_{suppf} f d\mu_+ - \int_{suppf} f d\mu_-.$$

Small caveat with the topology of $C_c(\Omega)$, $f_n \to f$ if $\|f_n - f\|_\infty \to 0$ and support of $\bigcup_n suppf_n$ is compact. More, next class.

What happens to L^1

Corollary

As C_c with this topology is separable, then bounded sets of $\mathcal{M}_{loc}$ are weak- compact.*

A sequence $\{f_n\} \subset L^1$ such that $\sup_n \|f_n\|_{L^1} < \infty$, then, up to subsequence, there exists $\mu \in \mathcal{M}(\Omega)$ such that

$$\int \varphi f_n \to \int \varphi d\mu$$

for every $\varphi \in C_b$.

μ is not necessarily in L^1!

We either say $f_n \rightharpoonup^* \mu$ or $f_n \rightharpoonup^{\mathcal{D}'} \mu$ or just weakly.

Measures

Let X be a set and Σ a σ-algebra over X. A function μ from Σ to the $[0, \infty]$ is called a **measure** if it satisfies the following properties:

- **Non-negativity**
- **Null empty set**
- σ**-additivity**

Radon measures are the measures such that Σ is the Borel σ-algebra. i.e. the minimal σ-algebra that contains open sets.

The space of test functions

Given $\Omega \subset \mathbb{R}^n$, we consider the topological vector space $C_c^\infty(\Omega)$, with the following notion of convergence:
Given $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$, we say that

$$\varphi_n \to^{\mathcal{D}} \varphi,$$

if

- All the supports of φ_n are contained in the same compact set

$$\bigcup supp\varphi_n \subset K \subset \Omega$$

- φ_n and all its derivatives converge uniformly to φ:

$$\|D^\alpha \varphi_n - D^\alpha \varphi\|_\infty \to 0 \qquad \forall \alpha.$$

The space of test functions

This topology is not metrizable (it can not be induced by a distance), it is only a locally convex topological space.

The topology is quite strange so that the topology of the dual is well behaved.

This way we introduce the space of Distributions

$$\mathcal{D}'(\Omega) = (\mathcal{D}(\Omega))^*.$$

Distributions

Definition

$T : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a distribution if

- T is linear.
- For every $K \subset\subset \Omega$, exists $C(K) > 0$ and $N(k) \in \mathbb{N}$ such that

$$|T(\varphi)| \leq C \sup_{|\alpha| \leq N} \|D^\alpha \varphi\|_\infty \qquad \forall \varphi \in C_c^\infty(K).$$

Remark: We say that a distribution T is of order N_0 if the same N_0 serves for all compact sets: $T \in \mathcal{D}'_{N_0}(\Omega)$.

Distributions

Why do we care about distributions:

- The space C_c^∞ is one of the tiniest spaces we can consider, hence it's dual is huge.
- In the same spirit as measures, they have good compactness properties.

Distributions

It contains all locally finite measures (signed radon measures) and it's distributional derivatives: Given $\mu \in \mathcal{M}_{loc}(\Omega)$ we have

$$\langle T_\mu, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \int_\Omega \varphi \, d\mu`$$

is a distribution.

Also, we can take derivatives

$$\langle D^\alpha T_\mu, \varphi \rangle = (-1)^{|\alpha|} \int_\Omega D^\alpha \varphi d\mu.$$

Theorem

$$\mathcal{D}'_0(\Omega) = \mathcal{M}_{loc}(\Omega)$$

Distributions

Another useful Theorem

Theorem (Riesz-Representation)

A distribution T is positive, i.e. for every $\varphi \geq 0$

$$\langle T, \varphi \rangle \geq 0$$

if and only if, it is induced by a positive Radon measure $\mu \in \mathcal{M}_+(\Omega)$.

$$T = T_\mu.$$

Distributions

We can take $\mu = \delta_{x_0}$ with $x_0 \in \Omega$, then

$$\langle T_\mu, \varphi \rangle_{\mathcal{D}', \mathcal{D}} = \varphi(x_0).$$

However,

$$T(\varphi) = \sum_{k=1}^{\infty} \frac{1}{k^2} \partial_1^k \varphi(x_0)$$

is not a distribution.

The support being a point determines it almost completely.

Distributions

We endow the space of Distributions $\mathcal{D}'$ with the weak-* topology associated as a dual to $\mathcal{D}$:

Definition

A sequence $T_j \rightharpoonup^{\mathcal{D}'} T$, iff and for every $\varphi \in \mathcal{D}$, we have

$$T_j(\varphi) = \langle T_j, \varphi \rangle_{\mathcal{D}',\mathcal{D}} \to \langle T, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = T(\varphi).$$

Remark: This is not well-behaved with a.e. convergence.

Distributions:

Given any continuous linear operation in $\mathcal{D}$, it can be extended by duality to $\mathcal{D}'$. For example differentiation is a bounded operator

$$\partial_1 : \mathcal{D} \to \mathcal{D}.$$

Therefore, abstractly we can define it's adjoint

$$\partial_1^* : \mathcal{D}' \to \mathcal{D}'.$$

i.e.

$$\langle \partial_1^* T, \varphi \rangle = \langle T, \partial_1 \varphi \rangle.$$

Distributions:

We have integration by parts for smooth functions:If T is induced by a smooth function f i.e.

$$\langle T, \partial_1 \varphi \rangle = \int_\Omega \partial_1 \varphi f \; dx = - \int_\Omega \varphi \partial_1 f \; dx = \langle \partial_1^* T, \varphi \rangle.$$

Therefore, $\partial_1^* T$ is induced by a $-\partial_1 f \in C^1(\Omega)$.

Distributions

Definition (Distributional Derivative)

$$D^\alpha T(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi).$$

Remark: Derivatives commute, because they commute for C^∞ functions.

Distributions

Examples:

- The derivative of the Heavyside function is the delta.
- We can take finite derivatives of a delta.

Theorem

If a distribution T is supported on a point x_0, then it there exists $N \in \mathbb{N}$ such that

$$T(\varphi) = \sum_{|\alpha|<N} c_\alpha D^\alpha \varphi(x_0).$$

Distributions

We can do this trick also for:

- Translations $(\tau_v T)(\varphi) = T(\tau_{-v}(\varphi))$, for $v \in \mathbb{R}^n$ small enough.
- We can take difference quotients of distributions

$$\frac{\tau_{he_1} T - T}{h} \to_{h\to 0}^{\mathcal{D}'} \partial_1 T$$

- Multiplication by a smooth function: $(\psi T)(\varphi) = T((\psi\varphi))$.

Distributions

Convolution: given $g \in \mathcal{D}(\mathbb{R}^n)$ and $T \in \mathcal{D}'(\mathbb{R}^n)$:

$$(g * T)(\varphi) = T(\varphi *_- g).$$

This smoothes out the distribution to become a C^∞ function!It is the same principle as mollification

$$D^\alpha(g * T) = (D^\alpha g) * T = g * (D^\alpha T) \in C^\infty.$$

In fact

$$(g * T)(x) = T(g(x - \cdot)) = T((\tau_x g)_-).$$

Distributions

This can be extended for non-smooth objects, if the support of one of the distributions is compact.

Definition

$$suppT = \{x \in \mathbb{R}^n \ : \ \text{for all neighborhood } U \text{ of } x \text{ exists } \varphi \in C_c^\infty(U)$$
$$\text{such that } T(\varphi) \neq 0\}.$$

Given T and S, such that one is with compact support we define

$$(T * S)(\varphi) = T(S *_{-} \varphi),$$

and it is continuous on T and S.

Distributions

We have an identity in this algebra, which is the delta at zero δ_0:

$$\delta_0 * T = T \qquad \forall T \in \mathcal{D}'(\mathbb{R}^n).$$

We can use this to show mollification converges to the distribution:

$$(g_\varepsilon * T) \to^{\mathcal{D}'} (\delta_0 * T) = T.$$

Distributions

Theorem

Given $\Omega \subset \mathbb{R}^n$, we can consider $g_\varepsilon * T|_{\Omega_\varepsilon} \in C^\infty(\Omega_\varepsilon)$ is the restriction of the convolution to the set

$$\Omega_\varepsilon = \{x \in \Omega \ : \ d(x, \partial\Omega) > \varepsilon\},$$

converges to T in distributions.

Remark: This is a function in Ω_ε

$$g_\varepsilon * T(x) = T(g_\varepsilon(x - \cdot)).$$

Proof.

Attempt. □

Distributions

We also know that classical derivatives and distributional derivatives match.

Theorem

A distribution T is in C^1 (i.e. exists $u \in C^1$ s.t. $T = T_u$), if and only if the partial derivatives $\partial_j T$ are induced by C^0 functions.

Proof.

The first implication follows by integration by parts.
The other implication, follows by mollifying the distribution and obtaining a Lipschitz bound to apply Arzela-Ascoli. Then use FTC and pass to the limit. Attempt! □

Distributions

We can define the derivatives for discontinuous objects, by employing correctly integration by parts.
First example

$$u(x) = \begin{cases} f(x) & x < x_0 \\ g(x) & x \geq x_0. \end{cases}$$

Then the distributional derivative is given by

$$u'(x) = f'(x)\chi_{(-\infty,x_0)} + g'(x)\chi_{(x_0,\infty)} + (g(x_0) - f(x_0)\delta_0)$$

Distributions

Theorem

Let $\Omega' \subset \Omega$ with C^1 (or some regular object)

$$u(x) = \begin{cases} f(x) & x \in \Omega' \\ g(x) & x \in \Omega \setminus \Omega'. \end{cases}$$

Then

$$\partial_j u = \partial_j f(x) \chi_{\Omega'} + \partial_j g(x) \chi_{\Omega \setminus \Omega'} + (g - f) \overline{n}_j \mathcal{H}^{n-1}|_{\partial \Omega' \cap \Omega},$$

where $\overline{n}$ is the outward normal of Ω'.

Distributions

Proof.

Follows from the Gauss-Green/Divergence theorem formula

$$\int_U \nabla \cdot (F)\, dx = \int_{\partial U} F \cdot \overline{n}\, dS$$

$\square$

Tempered Distributions

There is a highlighted set within distributions, which is called **tempered distributions** denoted by $\mathcal{S}'(\mathbb{R}^n)$ which is the dual of the Schwartz class $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.
This is the class of rapidly decreasing functions

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty : \sup_x \left||x|^k D^\alpha f(x)\right| < \infty \forall \alpha \ \& \ k \in \mathbb{N}\}$$

Tempered Distributions

The main point to define this class is that it is preserved under the Fourier transform:

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

$$\mathcal{F}(\varphi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \varphi(x)\, dx$$

Tempered Distributions

By duality we can extend it to $\mathcal{S}' \subset \mathcal{D}'$.

By mollification arguments we notice that it coincides with the traditional version, when the Fourier transform is classically defined.

One of the main properties is that

$$\mathcal{F}(D^\alpha T) = \xi^\alpha \mathcal{F}(T).$$

Remark: We can measure regularity this way.

Theorem

Every linear PDE has a distributional solution!

Proof.

Attempt. □