Functional Analysis

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Who is this guy teaching me Functional?

Main field of study: PDEs arising from modelling, in particular Stochastic interacting particle systems and Wasserstein gradient flows. That's why most of the applications will be in PDEs.

Bio:

2006-2011	Lic. en Matematica at UNC (Cordoba, Argentina)
2011-2016	Ph.D. Applied Mathematics at UMD (Maryland, United States)
2016-2017	Postdoc at ICTP (Trieste, Italy)
2017-2019	Postdoc at Imperial College (London, England)
2019-	Proffessor at PUC (Rio de Janeiro, Brazil)
2020-2022	Hooke Fellow at Oxford (Oxford, England)

Outline for the course

- Differential Calculus and L^p spaces.
- Distributions.
- Sobolev Spaces.
- Embedding Theorems.

Recomended books

Foundations functional analysis and distributions

- Linear Functional Analysis by Rynne, Bryan, Youngson, M.A.
- Functional analysis by Walter Rudin

Sobolev Spaces

- Measure Theory and Fine Properties of Functions, by L.C. Evans and R.F. Gariepy.
- Functional Analysis, Sobolev Spaces and Partial Differential Equations, by Haim Brezis
- Sobolev Spaces, by R.A. Adams and J.J.F. Fournier.
- The Analysis of Partial Differential Operators I, by L. Hörmander.

Not so gentle introduction

Functional Analysis: When Analysis/(point set) Topology meets Linear Algebra. The study of the Topology of infinite dimensional of (mostly linear) spaces.
Given a set U (e.g. $U \subset \mathbb{R}^n$), the set of scalar functions $F(U) = \{f : f : U \to \mathbb{R}\}$ has a clear vector space structure.

That is to say:

- If $f, g \in F(U)$, then $f + g \in F(U)$, where $(f + g)(x) = f(x) + g(x)$.
- If $f \in F(U)$ and $c \in \mathbb{R}$, then $(cf) \in F(U)$, where $(cf)(x) = cf(x)$.

Not so gentle introduction

Definition

A subspace $X \subset F(U)$ endowed with topology τ, is a Topological Vector Space. If the operations addition and multiplication by a scalar are continuous in the topology τ.

More specifically, we consider summing two functions as an application $sum : F(U) \times F(U) \to F(U)$. Then, the definition is asking that *sum* is continuous when we endow $F(U) \times F(U)$ with the product topology $\tau \times \tau$.

i.e. whenever we have sequence $\{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}} \subset X$, such that

$$f_n \to^{\tau} f \qquad \& \qquad g_n \to^{\tau} g,$$

then

$$f_n + g_n \to^{\tau} f + g.$$

Not so gentle introduction

The most studied types of vector spaces:

- **Metrizable/Metric:** There exists a metric/distance $d_\tau(\cdot,\cdot) : X \times X \to [0,\infty)$, satisfying the triangle inequality that induces the topology τ.

- **Normed:** There exists a norm $\|\cdot\|_\tau : X \to [0,\infty)$ that induces the metric/distance $d_\tau(f,g) = \|f-g\|_\tau$ that induces the topology τ.

- **Inner Product:** There exists an inner product $\langle \cdot,\cdot\rangle_\tau : X \times X \to \mathbb{R}$ that induces a norm $\langle f,f\rangle_\tau^{1/2} = \|f\|_\tau$ that induces a metric, that induces the topology τ.

Not so gentle introduction

Examples: Take $U = \mathbb{R}^n$ or any other measure space with underlying measure μ, we have

$$L^p(U) = \{f \in F(U) \cap \mathcal{M}(U) \ : \ \int_U |f(x)|^p \, d\mu(x) < \infty\}.$$

- For $p \geq 1$, L^p is a **Normed** vector space.
- For $p = 2$, L^2 is an **Inner-Product** space.
- For $0 < p < 1$, L^p is a **Metric** vector space with distance

$$d_p(f, g) = \int_U |f - g|^p \, d\mu.$$

Not so gentle introduction

Definition

A **metric** space X is said to be **complete** if every Cauchy sequence admits a limit.

Reminder: $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, if for every $\epsilon > 0$ exists $N(\epsilon) \in \mathbb{N}$ such that if $n, m \geq N(\epsilon)$, then

$$d(f_n, f_m) < \epsilon.$$

By the definition there exists a unique $f_\infty \in X$, such that

$$\lim_{n\to\infty} d(f_n, f_\infty) = 0.$$

Not so gentle introduction

Completeness is such an important property that spaces change their name.

- Complete metric spaces are called **Frechet** Spaces (or F-Spaces depending if the balls can be taken to be convex).
- Complete normed spaces are called **Banach** Spaces.
- Complete inner product spaces are called **Hilbert** Spaces.

Not so gentle introduction

Hilbert spaces are the most similar to finite dimensional vector spaces.They admit an infinite complete Orthonormal basis $\{e_i\}_{i\in I}$ such that

$$\langle e_i, e_j\rangle = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and simplifies the proofs greatly.

Warning: The set of indexes I is not necessarily countable!

Not so gentle introduction

Definition

A topological space X is said to be **separable** if there exists a countable set of points $\{f_n\} \subset X$ which is dense in X.

Theorem

If X is a **separable Hilbert** *space, then it admits a countable orthonormal basis.*

Corollary

If X is a **separable Hilbert** *space, then it is* **homeomorphic** *to $L^2([0,1])$.*

Not so gentle introduction

One of the main difference between finite dimensions and infinite dimensions is the compactness.

Theorem (Heine-Borel)

Any bounded sequence, admits a convergent subsequence. Alternatively, every closed bounded set is compact.

Corollary

In $\mathbb{R}^n$ *all norms induce the same topology.*

Not so gentle introduction

Infinite dimensional separable **Hilbert** spaces do not satisfy the **Heine-Borel** property.Take $\{e_n\}_{n\in\mathbb{N}}$, the orthonormal basis then

$$\|e_i - e_j\|^2 = \langle e_i - e_j, e_i - e_j\rangle = 2(1 - \langle e_i, e_j\rangle) = 2\delta_{i,j}$$

For $L^2([0,1])$ this is like taking the Fourier basis

$$e_n = \sin(2\pi nx) \rightharpoonup 0$$

This is known as the Riemann-Lebesgue Lemma.

Differential Calculus

Given two **normed** vector spaces X, Y we can consider $\Omega \subset X$ open, an application

$$F : \Omega \subset X \to Y$$

and the set of continuous linear mappings

$$\mathcal{L}(X, Y) = \{L : X \to Y \ : \ L \text{ is linear and continuous}\}.$$

Theorem

If X and Y are normed spaces then $\mathcal{L}(X, Y)$ is normed space.

If $Y = \mathbb{R}$ with the usual topology, then we denote $\mathcal{L}(X, R) = X^*$.

Differential Calculus

Definition

A mapping $F : \Omega \to Y$ is said to be (Frechet) differentiable at $x_0 \in \Omega$ if exists $L \in \mathcal{L}(X, Y)$ such that

$$F(x) = F(x_0) + L(x - x_0) + o(|x - x_0|).$$

Remark: L is unique and we denote $DF(x_0) = dF(x_0) = L$.

Remark: If F is differentiable at $x_0 \in \Omega$, then we say that it is differentiable in Ω.

Differential Calculus

If F is differentiable in Ω we can consider the mapping $DF : \Omega \subset X \to \mathcal{L}(X, Y)$ and ask if it is Frechet Differentiable.If it is differentiable and it's differential is continuous then we say that $F \in C^2(\Omega; Y)$, and

$$D^2F \in \mathcal{L}(X^2, Y).$$

The idea is to lose fear and realize that most of the properties that you know for differentiable vector valued functions $F : \mathbb{R}^n \to \mathbb{R}^m$ are still valid in this case.

Differential Calculus

For instance,

Theorem (Fundamental Theorem of Calculus)

If $F : (a, b) \subset \mathbb{R} \to Y$ is C^1 and Y is a **Banach** *space, then*

$$F(t) = F(s) + \int_s^t dg(u)\, du,$$

where we define the integral via Riemman sums.

Remark: Mean Value theorem doesn't hold, but MV Inequality does.

Differential Calculus

A vector valued function is C^1, if and only if, it has continuous partial derivatives.

Theorem

Let $D \subset \mathcal{S}_X = \{x : \|x\| = 1\}$, such that $cl(Span(D)) = X$, then $F \in C^1$ if and only if

- *F is continuous.*
- *For every $x \in \Omega$ $F(x + \cdot d) : \mathbb{R} \to \mathbb{R}$ is differentiable.*
- *There exists $g : \Omega \to \mathcal{L}(X, Y)$ continuous such that*

$$\frac{d}{dt} F(x + td) = g(x + td)(d) \in Y.$$

In this case $dF = g$.

L^p spaces

Given a measure space $(\Omega, \mathcal{F}, \mu)$ we can define

$$L^p(\Omega; d\mu) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |f|^p \, d\mu < \infty \right\}.$$

Reminder: Measurable $f^{-1}((a,b)) \in \mathcal{F}$ for all a, b.They are defined almost everywhere (a.e.), with respect to the measure

$$f \sim g \qquad \text{if and only if } \mu(\{f \neq g\}) = 0.$$

For instance we say $f \in L^p$ is continuous if there exists a continuous representative.

Typical case $\Omega \subset \mathbb{R}^n$ and μ is the Lebesgue measure.

L^p spaces

L^p endowed with the norm

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu \right)^{1/p}$$

is a complete normed vector space. L^p is a Banach space!

- For $1 \leq p < \infty$ it is separable. Ex: $L^\infty((0,1))$ is not separable.
- For $1 \leq p < \infty$, the dual of L^p is $(L^p)^* = L^q$, where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

- For $1 < p < \infty$ L^p is reflexive

$$(L^p)^{**} = L^p.$$

L^p spaces

Usefuel Inequalities:

- Hölder inequality or duality pairing for $X = L^p(\Omega; d\mu)$

$$\left|\int_\Omega fg\ d\mu\right| = |\langle f, g\rangle_{X,X^*}| \leq \|f\|_X \|g\|_{X^*} = \|f\|_{L^p}\|g\|_{L^q},$$

  if $1/p + 1/q = 1$
  A useful variation

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q},$$

  if $1/p + 1/q = 1/r$.

L^p spaces

- Minkowski's or triangle inequality for the norm

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

- Interpolation

$$\|f\|_{L^r} = \|f\|_{L^p}^{\lambda}\|f\|_{L^q}^{1-\lambda}$$

if

$$\frac{\lambda}{p} + \frac{1-\lambda}{q} = \frac{1}{r}.$$

L^p spaces

- Young's inequality for the convolution

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$$

if

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Mollification

From now on $\Omega \subset \mathbb{R}^n$ open.
We consider the test functions

$$C_c^\infty(\Omega) = \{\varphi \in C_c^\infty \ : \ supp\varphi = \{x \in \Omega \ : \ \varphi(x) \neq 0\} \text{ is compact}\}.$$

When endowed with a specific topology they are denoted by

$$\mathcal{D}(\Omega).$$

The basic building block is

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

Mollification

We can show existence of partitions of unity

Lemma

Given an open covering $\{U_i\}_{i\in I}$ *there exists positive functions* $\varphi_i \in C_c^\infty(U_i)$ *such that*

$$\sum_{i\in I} \varphi_i(x) = 1 \qquad \forall x \in \Omega,$$

such that only a finite number of them are non-zero.

Mollification

We can show existence of cut-off functions with estimates

Lemma

For any V compact subset Ω, there exists $\chi_V \in C_c^\infty(\Omega)$, such that

$$\chi_V(x) = 1 \qquad \text{for all } x \in V$$

and

$$|D^\alpha(\chi_V)(x)| \leq C_\alpha d(x, \partial\Omega)^{-|\alpha|}.$$

Mollification

To mollify a function we need to take a $\varphi \in C_c^\infty(B_1)$ which is positive and

$$\int_{B_1} \varphi = 1.$$

We denote by

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\frac{x}{\varepsilon}).$$

A mollification of a function $f \in L^p$ is given by

$$f^\varepsilon(x) = f * \varphi_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\varphi_\varepsilon(x-y)\, dy,$$

where we have extended f by zero outside of Ω.

Mollification

Properties:

- $f^\varepsilon \in C^\infty$ and
$$D^\alpha f^\varepsilon = (D^\alpha \varphi_\varepsilon) * f.$$
- By Young's
$$\|f^\varepsilon\|_{L^p} \leq \|f\|_{L^p} \|\varphi_\varepsilon\|_{L^1} = \|f\|_{L^p}.$$
- It approximates f in L^p:
$$\|f^\varepsilon - f\|_{L^p} \to 0$$
- $suppf^\varepsilon \subset (suppf) + B_\varepsilon$.

Then C_c^∞ dense in L^p.

Convergence Theorems

Theorem (Fatou's Lemma)

If $f_n \to f$ a.e and $f_n \geq 0$, then

$$\liminf \int f_n \geq \int \liminf f_n = \int f.$$

Theorem (Monotone Convergence)

If $f_n \to f$ a.e. is monotone increasing i.e. $f_n(x) \leq f_{n+1}(x)$ for every x and n, then

$$\lim_n \int f_n = \sup_n \int f_n = \int \sup_n f_n = \int f.$$

Convergence Theorems

Theorem (Lebesgue Dominated Convergence)

If $f_n \to f$ *a.e. and exists* $g \in L^1$ *such that* $|f_n|(x) \leq g(x)$ *for every* n *and* x *then*

$$\lim \int f_n = \int \lim f_n = \int f.$$

Convergence Theorems

Theorem (Vitali's Convergence theorem)

Given Ω a set of finite measure $|\Omega| < \infty$, then $f_n \to f$ in L^p, i.e.

$$\|f_n - f\|_{L^p} \to 0,$$

if and only if,

- $f_n \to f$ *in measure, i.e. for every* $\varepsilon > 0$

$$|\{x \ : \ |f_n(x) - f(x)| > \varepsilon\}| \to 0.$$

- *The family* $\{f_n\}$ *is equintegrable. i.e. for every* $\varepsilon > 0$ *exists a* δ *such that for every measurable set* A *satisfying* $|A| < \delta$ *implies*

$$\int_A |f_n|^p \leq \varepsilon.$$

Convergence/Compactness

Theorem (Riesz-Kolmogorov)

For $1 \leq p < \infty$, a family $\{f_i\}_{i\in I}$ is pre-compact in $L^p(\mathbb{R}^n)$, iff,

- *It is Bounded*
$$\sup_I \|f_i\|_{L^p} < \infty.$$

- *It is equi-continuous:*
$$\sup_I \|f_i(\cdot + h) - f_i(\cdot)\|_{L^p} \to 0.$$

- *It is tight: for every $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that*
$$\sup_I \int_{K^c} |f_i|^p \leq \varepsilon.$$

Convergence/Compactness

This should be reminiscent of compactness for continuous functions:

Theorem (Arzela-Ascoli)

A family of continuous functions $\{f_i\}_{i\in I}$ *from a compact set* K *is pre-compact, iff,*

- *They are uniformly bounded*

$$\sup_I \|f_i\|_\infty < \infty$$

- *They have a uniform modulus of continuity. i.e. there exists* $\omega : [0,\infty) \to [0,\infty)$ *such that* ω *is increasing and*

$$\sup_{x,y,I} d(f_i(x), f_i(y)) \leq \omega(d(x,y)).$$

Convergence/Compactness

The main point is that there is three necessary and sufficient conditions for convergence:

- **Boundedness** in the appropriate norm.
- Uniform **Regularity** measured in the appropriate norm.
- **Tightness**, you need be careful things are not escaping to infinity.

Weak vs Weak-* convergence

Given a space X, the weak topology is the smallest/coarsest topology that makes every element of X^* continuous.i.e.

$$f_n \rightharpoonup f \qquad \text{iff} \qquad \langle f_n, T\rangle_{X,X^*} \to \langle f, T\rangle_{X,X^*} \qquad \forall T \in X^*$$

Given X^*, the weak-* topology is the smallest/coarsets topology that makes the elements of $X \subset X^{**}$ continuous.i.e.

$$T_n \rightharpoonup^* T \qquad \text{iff} \qquad \langle f, T_n\rangle_{X,X^*} \to \langle f, T\rangle_{X,X^*} \qquad \forall f \in X.$$

Weak vs Weak-* Convergnce

Theorem (Banach-Alaoglu)

If X is separable, then bounded sets of X^* are compact with the weak- topology.*

Remark: When $X = X^{**}$ is reflexive, then the weak topology is equal to weak-* on X^*.

Corollary

For $1 < p < \infty$ bounded sets are weakly compact.

Corollary

For $p \in L^\infty$ bounded sets are weak- compact.*

Bounded sets in L^1 are not compact!

Strong vs Weak convergence

There a few ways weakly convergent sequences can fail to converge strongly:

- **Concentration:**

$$f_n(x) = n^{p/n} f(nx)$$

- **Oscillation:**

$$f_n(x) = sin(nx) f(x).$$

- **Not tight:**

$$f_n(x) = \frac{1}{n^{1/p}} \chi_{(-n,n)}$$

Weak convergence

Weak or Weak-* convergence can be characterized by any dense subset of the space X or X^*.

Theorem (Dunford-Pettis)

$|\Omega| < 1$ and $1 < p < \infty$, then $f_n \rightharpoonup f$, if and only if

- $$\int_Q f_n \to \int_Q f \qquad \textit{for all cubes } Q$$
- $$\sup_n \|f_n\|_{L^p} < \infty.$$

Remark: For $p = \infty$, replace weak by weak-*. For $p = 1$, we need equiintegrability.

What happens to L^1?

$$L^1(\Omega) \subset \mathcal{M}(\Omega),$$

where $\mathcal{M}(\Omega)$ is the set of signed bounded Radon measures.

$$\mu \in \mathcal{M}(\Omega) \qquad \text{iff} \qquad \mu = \mu_+ - \mu_- \qquad \mu_+, \mu_- \in \mathcal{M}_+(\Omega),$$

where $\mathcal{M}_+(\Omega)$ is the set of bounded Radon measures.
Given $f \in L^1(\Omega)$, we can consider

$$\mu_f = f d\mathcal{L} \qquad \text{i.e.} \qquad \mu_f(A) = \int_A f \, d\mathcal{L}.$$

What happens to L^1

Theorem (Riesz-Representation Theorem)

The dual of continuous functions with compact support is locally finite measures:

$$\mathcal{M}_{loc}(\Omega) = (C_c(\Omega))^*.$$

i.e. for every $T \in (C_c(\Omega))^*$ there exists a unique $\mu \in \mathcal{M}_{loc}(\Omega)$ such that

$$\langle f, T \rangle_{C_c(\Omega),(C_c(\Omega))^*} = \int_{suppf} f \, d\mu = \int_{suppf} f d\mu_+ - \int_{suppf} f d\mu_-.$$

Small caveat with the topology of $C_c(\Omega)$, $f_n \to f$ if $\|f_n - f\|_\infty \to 0$ and support of $\bigcup_n suppf_n$ is compact. More, next class.

What happens to L^1

Corollary

As C_c or C_b are separable, then bounded sets of $\mathcal{M}$ or $\mathcal{M}_{loc}$ are weak- compact.*

Finally, consider $\{f_n\} \subset L^1$ such that $\sup_n \|f_n\|_{L^1} < \infty$, then, up to subsequence, there exists $\mu \in \mathcal{M}(\Omega)$ such that

$$\int \varphi f_n \to \int \varphi d\mu$$

for every $\varphi \in C_b$.μ is not necessarilly in L^1.