# Alexandrov's Theorem revisited

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Joint work with Francesco Maggi

# Capillarity Model

We consider $\Omega \subset \mathbb{R}^{n+1}_+$ a droplet, the associated free energy is given by

$$\mathcal{F}(\Omega) \quad = \quad P(\Omega; \mathbb{R}^{n+1}_+) \quad + \quad \int_W \sigma(z)\, dz \quad + \quad \int_\Omega g(y)\, dy$$

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## Question

*What does the energy landscape look like? Why do we only see spherical caps?*

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We will focus only on the energy landscape of perimeter, but similar results seem to hold for the constant adhesion coefficient case. (work in progress)

# Isoperimetric problem

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# Sets of finite perimeter

Perimeter has been extended by lower-semicontinuity to any $\Omega \subset \mathbb{R}^{n+1}$:

$$P(\Omega) = \inf\left\{\liminf_{n\to\infty} P(\Omega_n) \;:\; \{\Omega_n\}_{n\in\mathbb{N}} \text{ smooth} \quad \& \quad \lim_{n\to\infty} |\Omega_n \triangle \Omega| = 0\right\}.$$

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$\Omega$ is a set of finite perimeter (SOFP), if $P(\Omega) < \infty$.

This is the natural competition class.

# How bad can a SOFP be?

If $\Omega$ is a SOFP, then there exists $\partial^*\Omega \subset \partial\Omega$ such that for every point in $x \in \partial^*\Omega$ there exists measure theoretical unit normal $\nu_\Omega(x)$:

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$$\int_\Omega \mathrm{div}\,(X) = \int_{\partial^*\Omega} X \cdot \nu_\Omega \qquad \forall X \in [C_c^1(\mathbb{R}^{n+1})]^{n+1}$$

# Pathological Example

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$\Omega_\infty$ is a set of finite perimeter.

# Stability of the global minimizer

**Theorem (Fusco, Maggi, Pratelli)**

*If* $|\Omega| = |B_1|$, *then there exists* $c(n) > 0$ *such that, up to translation,*

$$c|\Omega \triangle B_1|^2 \leq P(\Omega) - P(B_1).$$

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Are there any local minimizers or critical points?

# Euler-Lagrange conditions

For any $X \in [C_c^1(\mathbb{R}^{n+1})]^{n+1}$ there is a family of diffeomorphisms $T_t(x) = x + tX(x) + O(|t|^2)$.

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Taking first variations

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where $H_{\partial\Omega}$ is the distributional mean curvature of the boundary.

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Hence, the mean curvature is constant $H_{\partial\Omega} = \lambda = \frac{nP(\Omega)}{(n+1)|\Omega|}$.

## Mean curvature refresher

- In differential geometry

$$H_{\partial\Omega} = trace(II_{\partial\Omega}) = \sum_{i=1}^{n} \kappa_i,$$

where $\kappa_i$ are the principal curvatures.

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- In PDE's, if $\partial\Omega$ is locally the graph of $u$, then

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$$H_{\partial\Omega} = -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$
We have strong comparison principle and regularity for Lipschitz weak solutions!
- This also comes out naturally of the first variation of perimeter. Again, if $\partial\Omega$ is locally the graph of $u$, then locally
$$P(\Omega) = \int \sqrt{1+|\nabla u|^2},$$
and the expression for the first variation follows.

# Alexandrov's Theorem 1962

## Theorem (Alexandrov)

*Let $\Omega \subset \mathbb{R}^{n+1}$ be a connected open bounded set with $C^2$ boundary. If $H_\Omega$ is constant, then $\Omega$ is a ball.*

# Alexandrov's Theorem 1962

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### Proof.

Moving planes method. $\square$

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Quantitative stability for smooth almost critical points. (Ciraolo-Maggi for $\|H_\Omega - \lambda\|_{L^\infty(\partial\Omega)}$, D-Maggi-Mihaila-Neumayer $\|H_\Omega - \lambda\|_{L^2(\partial\Omega)}$ for smooth anisotropies)

# Heintze-Karcher inequality

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Let $\Omega$ be a smooth set with positive mean curvature, then

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The proof can be done by shooting rays of length $\frac{n}{H}$ from the boundary and covering $\Omega$.

# Montiel-Ros's proof

Set up:

$$g(x,t) = x - t\nu_\Omega, \qquad \Gamma = \{(x,t)\ :\ x \in \partial\Omega,\ 0 < t < \kappa_n^{-1}(x)\}.$$

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This is an alternative proof of Alexandrov's theorem.

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**Lemma**

*Let $\Omega$ a SOFP be a critical point of perimeter at fixed volume. Up to a modification of measure zero, the singular set $\Sigma = \partial\Omega \setminus \partial^*\Omega$ satisfies $\mathcal{H}^n(\Sigma) = 0$ and $\partial^*\Omega$ is locally analytic.*

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## Lemma

*Let $\Omega$ a SOFP be a critical point of perimeter at fixed volume. Up to a modification of measure zero, the singular set $\Sigma = \partial\Omega \setminus \partial^*\Omega$ satisfies $\mathcal{H}^n(\Sigma) = 0$ and $\partial^*\Omega$ is locally analytic.*

## Proof.

Monotonicity fomula and Allard's regulalrity. $\square$

# Step 1 revisited

New set up

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Proof by contradiction.

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**Observation**:
Given $y_1$, $y_2 \in \Omega$. If there exists $x \in \partial\Omega$ such that $d(y_1, \partial\Omega) = |x - y_1|$ and $d(y_2, \partial\Omega) = |x - y_2|$. Then $x$, $y_1$ and $y_2$ lie on a straight line.

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**Proof:**
Blow up. There are no stationary cones in a wedge.

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This finishes the proof in the local minimizer case by density estimates.

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**Observation:** For a.e. $t > 0$, the level sets of the distance function

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are $C^{1,1}$ rectifiable.

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**Conclusion:** For any $r \in [0, t]$, the map

$$\zeta_r = x + r\nu^{\Gamma^t}(x)$$

is Lipschitz on $\Gamma^t$.

# Step 1 revisited

**Combining observations:**

$$0 = 2\mathcal{H}^n(\Sigma) \geq \int_\Sigma \mathcal{H}^0(\zeta_t^{-1}(y)) = \int_{\Gamma^t \cap \zeta_t^{-1}(\Sigma)} J^{\Gamma^t}\zeta^t = \int_{\Gamma^t \cap \zeta_t^{-1}(\Sigma)} \prod_{i=1}^n (1+t\kappa_i).$$

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**Conclusion:**

$$\kappa_1^{\Gamma^t} = -\frac{1}{t} \qquad \text{a.e. on } \zeta_t^{-1}(\Sigma) \cap \Gamma^t.$$

# Step 1 revisited

Taking $s \in (0, t)$

$$\kappa_i \leq \frac{1}{t-s} \;\&\; \kappa_1 = -\frac{1}{s} \qquad \text{a.e. on } \zeta_{t-s}(\zeta_t^{-1}(\Sigma)) \subset \Gamma^s.$$

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There exists $y_0 \in \Gamma^s$, such that

$$H^{\Gamma^s}(y_0) = -1,$$

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We need a comparisson principle between a paraboloid and a varifold.

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**Lemma (Schätzle)**

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We reached a contradiction!

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**Same computation.**

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What ties them together?

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We show by contradiction

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The proof is done.

# Corollary

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*If* $\lim_{i\to\infty} |\Omega \triangle \Omega_i| = 0$, $\lim_{i\to\infty} P(\Omega_i) = P(\Omega)$ *and there exists* $\lambda$ *such that for every* $X \in [C_c^\infty(\mathbb{R}^{n+1})]^{n+1}$

$$\lim_{i\to\infty} \int_{\partial^*\Omega_i} \operatorname{div}^{\partial^*\Omega_i} X - \lambda X \cdot \nu = 0,$$

*then* $\Omega$ *is a finite union of balls of radius* $n/\lambda$.

# Possible Application

**Volume preserving mean curvature flow:**
Velocity field on $\partial\Omega(t)$

$$X(x,t) = -\left( H_{\partial\Omega(t)}(x) - \frac{1}{P(\Omega(t))} \int_{\partial\Omega(t)} H_{\partial\Omega(t)} \right) \nu^{\Omega(t)}.$$

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**Dissipation Inequality**

$$\frac{dP(\Omega(t))}{dt} = -\int_{\partial\Omega(t)} \left| H_{\partial\Omega(t)} - \frac{1}{P(\Omega(t))} \int_{\partial\Omega(t)} H_{\partial\Omega(t)} \right|^2.$$

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**Open question:** $\Omega(\infty)$ finite union of balls in general?

Thank you, any questions?